Entropy of SIC probability distributions for qubits and qutrits
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Using a symmetric informationally complete (SIC) measurement, we can map d-dimensional density operators to probability distributions in a (d^2-1)-dimensional simplex in the following way:

$$\rho = \sum_{j=1}^{d^2} \left[(d+1)p_i - \frac{1}{d} \right] \Pi_i, \tag{1}$$

where $\{\Pi_i\}_{i=1}^{d^2}$ denotes a set of rank-1 projections such that

$$\mathrm{Tr}\,(\Pi_i \Pi_j) = \frac{d\delta_{ij}+1}{d+1}, \qquad \frac{1}{d}\sum_{i=1}^{d^2} \Pi_i = I_d. \tag{2}$$

Since this is a bijective mapping, we get a representation for quantum states in terms of the probability distributions $\vec{p}$, with elements given by

$$p_i = \frac{1}{d}\mathrm{Tr}\,(\rho \Pi_i)\,. \tag{3}$$

A remarkable theorem tells us that a pure quantum state is given by

$$\mathrm{Tr}(\rho) = \mathrm{Tr}(\rho^2) = \mathrm{Tr}(\rho^3) = 1. \tag{4}$$

In the equivalent representation, the conditions for pure states are

$$\sum_{i=1}^{d^2} p_i^2 = \frac{2}{d(d+1)}, \qquad \sum_{i,j,k=1}^{d^2} T_{ijk} p_i p_j p_k = \frac{d+7}{(d+1)^3}. \tag{5}$$

Observe that the SIC projections also represent pure states. The probability distributions $\vec{q}_j, j = 1, 2, \ldots, d^2$ corresponding to SIC projections are called basis distributions and they are given by

$$q_{j,i} = \frac{d\delta_{ij}+1}{d(d+1)} \tag{6}$$

With probability distributions as quantum states, we can consider the following questions:

1. Which pure states yield the minimum Shannon entropy?
2. Which pure states yield the maximum relative entropy with respect to basis distributions.

In this note, we answer both questions for qubits and qutrits.
First, for qubits we minimize the Shannon entropy

$$H(\vec{p}) = -\sum_{i=1}^{4} p_i \log p_i \tag{7}$$

subject to the constraints

$$\sum_{i=1}^{4} p_i = 1, \qquad \sum_{i=1}^{4} p_i^2 = \frac{1}{3}, \tag{8}$$

which are pure-state conditions for qubits.

Using the method of Lagrange multipiers, we wish to find the critical points for the Lagrangian

$$\mathcal{L} = -\sum_{i=1}^{4} p_i \log p_i + \lambda \left(\sum_{i=1}^{4} p_i - 1 \right) + \mu \left(\sum_{i=1}^{4} p_i^2 - \frac{1}{3} \right). \tag{9}$$

To find the critical points, we differentiate $\mathcal{L}$ with respect to p_i and set to the result to zero:

$$\frac{\partial \mathcal{L}}{\partial p_i} = -\log p_i - 1 + \lambda + 2\mu p_i = 0 \tag{10}$$

for $i = 1, 2, 3, 4$. We can use any pair of equations above to eliminate λ. Then we find that

$$\mu = \frac{\log p_i - \log p_j}{2(p_i - p_j)} \tag{11}$$

for any $i \neq j$. Such a condition can hold only when at least 3 of the p_is are equal.

If all p_is are the same then $p_i = \frac{1}{4}$. However in this case,

$$\sum_i p_i^2 = \frac{1}{4}, \tag{12}$$

which means this probability distribution does not correspond to a pure state.

If 3 of the p_is are the same then we can set $p_j = p$ and $p_i = p'$ for $i \neq j$. From the pure-state conditions, we get

$$p + 3p' = 1, \qquad p^2 + 3p'^2 = \frac{1}{3}. \tag{13}$$

Solving for p and p', we get $p = 0, p' = \frac{1}{3}$ and $p = \frac{1}{2}, p' = \frac{1}{6}$.

Let $\vec{r}_j$ denote the probability distribution with $p_j = 0, p_{i\neq j} = \frac{1}{3}$. Note that $p_j = \frac{1}{2}, p_{i\neq j} = \frac{1}{6}$ correspond to the basis distributions $\vec{q}_j$. Then

$$H(\vec{r}_j) = \log 3, \qquad H(\vec{q}_j) = \frac{1}{2}\log 12. \tag{14}$$

Thus, the $\vec{r}_j$s are candidate minimum entropy states.

To confirm, that $\vec{r}_j$s are indeed minimum points, respectively, we can compute the determinant of the bordered Hessian. The bordered Hessian for a function $f(\vec{x})$ with constraint $g(\vec{x}) = c$ is the matrix

$$B(f,g) = \begin{pmatrix} 0 & \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_1} \\ \frac{\partial g}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial f}{\partial x_1 \partial x_n} \\ \vdots & & \ddots & \\ \frac{\partial g}{\partial x_n} & \frac{\partial f}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}. \tag{15}$$

In this case, we have

$$\mathrm{Det} B(\vec{r}_j) = \frac{4}{3}(3 - 2\mu)^2 > 0, \tag{16}$$

thus, confirming that the $\vec{r}_j$s yield the minimum among pure states.

Now consider the relative entropy

$$D(\vec{p} \mid\mid \vec{q}_j) = \sum_{i=1}^{4} p_i \log \left(\frac{p_i}{q_{j,i}} \right) \tag{17}$$

subject to the constraints in Eq. (8).

In this case, the Lagrangian is

$$\mathcal{L} = -\sum_{i=1}^{4} p_i \log\left(\frac{p_i}{q_{j,i}}\right) + \lambda\left(\sum_{i=1}^{4} p_i - 1\right) + \mu\left(\sum_{i=1}^{4} p_i^2 - \frac{1}{3}\right). \tag{18}$$

Thus,

$$\frac{\partial \mathcal{L}}{\partial p_i} = \log\left(\frac{p_i}{q_{j,i}}\right) + 1 + \lambda + 2\mu p_i = 0 \tag{19}$$

for $i = 1, 2, 3, 4$.

Since $\vec{q}_j$ is a basis distribution, this means that 3 of the p_is above have the same equation, in particular, those with $i \neq j$. Thus, we can let $p_j = p, p_{i\neq j} = p'$.

This is identical to the solution for Shannon entropy so using the pure-state conditions, we again get $p = 0, p' = \frac{1}{3}$ and $p = \frac{1}{2}, p' = \frac{1}{6}$. In this case, we have

$$D(\vec{r}_j \mid\mid \vec{q}_j) = \log 2 \tag{20}$$

as the maximum.

Next, for qutrits, we consider the probability distributions obtained using the Hesse SIC, which is the Weyl-Heisenberg SIC with fiducial vector

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \tag{21}$$

The pure-state conditions for the Hesse SIC are

$$\sum_{i=1}^{9} p_i = 1, \qquad \sum_{i=1}^{4} p_i^2 = \frac{1}{6}, \qquad \sum_i p_i^3 = \sum_{(ijk)\in Q} p_i p_j p_k, \tag{22}$$

where Q refers to the set of indices that correspond to lines on the following affine plane of order 3:

$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6, \\ 7 & 8 & 9 \end{matrix} \tag{23}$$

that is, $(ijk) = (123), (147), (159), (168)$, and so on. Here we minimize the Shannon entropy

$$H(\vec{p}) = -\sum_{i=1}^{9} p_i \log p_i \tag{24}$$

with constraints given by Eq. (22).

The Lagrangian in this case is

$$\mathcal{L} = -\sum_{i=1}^{9} p_i \log p_i + \lambda\left(\sum_{i=1}^{9} p_i - 1\right) + \mu\left(\sum_{i=1}^{4} p_i^2 - \frac{1}{6}\right) + \nu\left(\sum_i p_i^3 - \frac{1}{2}\sum_{(ijk)\in Q} p_i p_j p_k\right). \tag{25}$$

The critical points for the Lagrangian are given by

$$\frac{\partial \mathcal{L}}{\partial p_i} = \log p_i + 1 + \lambda + 2\mu p_i + 3\nu\left(p_i^2 - \frac{1}{2}\sum_{(ijk)\in Q} p_j p_k\right) = 0 \tag{26}$$

for $i = 1, 2, \ldots, n$. Observe that the set of equations (26) are invariant when in the affine plane (23), we fix an index i then interchange the rows and columns that do not contain i. To illustrate, suppose we choose $i = 1$. Then if $(ijk) \in Q$ is chosen as lines from

$$\begin{matrix} 1 & 3 & 2 \\ 7 & 9 & 8, \\ 4 & 6 & 5 \end{matrix} \tag{27}$$

we get the same equations as Eq. (26). This particular transformation of the affine plane implies that the critical points must have a corresponding symmetry that preserves the affine lines. One way to impose this symmetry is choose

$$p_2 = p_3, \qquad p_4 = p_7, \qquad p_5 = p_9, \qquad p_6 = p_8, \tag{28}$$

which is suggested by the example chosen above. Another way to keep the equations unchanged is to interchange 3 of the lines that pass through an index i. Again, with $i = 1$, we can have

$$\begin{matrix} 1 & 2 & 3 \\ 6 & 4 & 5, \\ 8 & 9 & 7 \end{matrix} \tag{29}$$

which would suggest that

$$p_4 = p_5 = p_6, \qquad p_7 = p_8 = p_9. \tag{30}$$

In this case, the resulting critical points will have the form

$$\vec{p} = (a, b, c, d, d, d, d, d, d) \tag{31}$$

with constraints given by

$$\begin{aligned} a + b + c + d &= 1, \\ a^2 + b^2 + c^2 + d^2 &= \frac{1}{6}, \\ a^3 + b^3 + c^3 + d^3 &= 3\left[abc + 2d^3 + 3(a + b + c)d^2\right]. \end{aligned} \tag{32}$$

The solutions are

$$(a, b, c, d) \in \left\{ \left(0, 0, 0, \frac{1}{6}\right), \left(0, \frac{1}{4}, \frac{1}{4}, \frac{1}{12}\right), \left(\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{12}\right), \left(\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{12}\right), \left(\frac{1}{3}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}\right) \right\}. \tag{33}$$

It is easy to check that the Shannon entropies are $\log 6$ for the first solution, $(\log 4 + \log 12)/2$ for the middle three, and $(\log 3 + 2\log 12)/3$ for the last one. Since

$$\log 6 < \frac{\log 4 + \log 12}{2} < \frac{\log 3 + 2\log 12}{3}, \tag{34}$$

the first solution yields a minimum while the last one yields a maximum.

Furthermore, the Shannon entropy is the same irrespective of the order of elements. Therefore, we have as many minimum points as possible states with 3 zeroes and 6 equal entries. Because the pure-state solutions have the symmetry of the affine plane of (23), there are 12 possible minimum states, with the position of the zeroes determined by the affine lines.