Quantum channels as affine maps on SIC probability distributions

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Let the Kraus decomposition for a general (completely positive) map between density operators ρ, ρ' of the same dimension be given by

$$\Phi(\rho) = \sum_{i=1}^{r} A_i \rho A_i^\dagger = \rho' \tag{1}$$

where A_i are the Kraus operators. Note that if the map is trace preserving then

$$\sum_{i=1}^{r} A_i^\dagger A_i = I. \tag{2}$$

What we want is to get the corresponding description in terms of a linear or affine map on SIC probability vectors. For this, consider a SIC with d^2 elements Π_i with

$$\mathrm{Tr}\,(\Pi_i) = 1, \qquad \mathrm{Tr}\,(\Pi_i \Pi_j) = \frac{d\delta_{ij} + 1}{d+1}. \tag{3}$$

We can then write density operators in terms of the SIC projectors,

$$\rho = \sum_{i=1}^{d^2} \left[(d+1)p_i - \frac{1}{d} \right] \Pi_i. \tag{4}$$

Because SICs form a basis for the space of operators, we can do the same for the Kraus operators

$$A_i = \sum_{j=1}^{d^2} a_{ij} \Pi_j. \tag{5}$$

Two special classes of maps to consider are unital and trace-preserving maps. If Φ is unital then $\Phi(I) = I$. Translating in terms of SICs,

$$\begin{aligned} \sum_{i=1}^{r} A_i A_i^\dagger &= \sum_{i=1}^{r} \sum_{j,k=1}^{d^2} a_{ij} a_{ik}^* \Pi_j \Pi_k \\ &= \sum_{i=1}^{r} \sum_{j,k,l=1}^{d^2} a_{ij} a_{ik}^* S_{jkl} \Pi_l = I \end{aligned} \tag{6}$$

where we used the structure coefficients,

$$\Pi_j \Pi_k = \sum_{l=1}^{d^2} S_{jkl} \Pi_l. \tag{7}$$

Because

$$\frac{1}{d}\sum_{l=1}^{d^2}\Pi_l = I \tag{8}$$

for SICs, we obtain that for each l

$$\sum_{i=1}^{r}\sum_{j,k=1}^{d^2} a_{ij}a_{ik}^{*}S_{jkl} = \frac{1}{d}. \tag{9}$$

We can sum both sides of Eq. (9) over $l = 1, 2, \ldots d^2$ to get

$$\sum_{i=1}^{r}\sum_{j,k=1}^{d^2} a_{ij}a_{ik}\left(\frac{d\delta_{jk}+1}{d+1}\right) = d. \tag{10}$$

Observe that the trace-preserving condition in Eq.(2) reads

$$\sum_{i=1}^{r}\sum_{j,k=1}^{d^2} a_{ij}^{*}a_{ik}S_{jkl} = \frac{1}{d}. \tag{11}$$

Moreover, from $\mathrm{Tr}\,(\Phi(\rho)) = \mathrm{Tr}\,(\rho) = 1$, we get

$$\sum_{i=1}^{r}\sum_{j,k,l=1}^{d^2} a_{ij}q_k a_{il}^{*}T_{jkl} = 1 \tag{12}$$

where we used the triple products

$$T_{jkl} = \mathrm{Tr}\,(\Pi_j\Pi_k\Pi_l) = \frac{d}{d+1}\left[S_{jkl} + \frac{d\delta_{jk}+1}{d(d+1)}\right] \tag{13}$$

and

$$q_k = (d+1)p_k - \frac{1}{d}. \tag{14}$$

For a unital, trace-preserving map, we can expand the left-hand-side of Eq. (12) and substitute Eq. (10) to obtain

$$\sum_{i=1}^{r}\sum_{j,k,l=1}^{d^2} a_{ij}p_k a_{il}^{*}T_{jkl} = \frac{2}{d+1}. \tag{15}$$

In general,

$$\Phi(\rho) = \sum_{i=1}^{r}\sum_{j,k,l} a_{ij}q_k a_{il}^{*}\Pi_j\Pi_k\Pi_l \tag{16}$$

Let $\Pi_j\Pi_k = \sum_m S_{jkm}\Pi_m$ and $\Pi_m\Pi_l = \sum_n S_{mln}\Pi_n$. Then

$$\Phi(\rho) = \sum_{i=1}^{r}\sum_{j,k,l=1}^{d^2}\sum_{m,n=1}^{d^2} a_{ij}q_k a_{il}^{*}S_{jkm}S_{mln}\Pi_n = \sum_{n=1}^{d^2}\left[\sum_{i=1}^{r}\sum_{j,k,l,m=1}^{d^2} a_{ij}q_k a_{il}^{*}S_{jkm}S_{mln}\right]\Pi_n. \tag{17}$$

Thus, Eq. (17) says that $\rho \mapsto \Phi(\rho)$ corresponds to the linear map

$$q_n \mapsto q'_n = \sum_{i=1}^{r} \sum_{j,k,l,m=1}^{d^2} a_{ij} q_k a^*_{il} S_{jkm} S_{mln}, \tag{18}$$

that is, each component of $\vec{q}$ given by Eq. (14) gets mapped to another vector given in terms of the expansion coefficients of the Kraus operators and the structure coefficients.

Writing it out explicitly, we have

$$(d+1)p'_n + \frac{1}{d} = (d+1) \sum_{i=1}^{r} \sum_{j,k,l,m=1}^{d^2} a_{ij} p_k a^*_{il} S_{jkm} S_{mln} - \frac{1}{d} \sum_{i=1}^{r} \sum_{j,k,l,m=1}^{d^2} a_{ij} a^*_{il} S_{jkm} S_{mln}. \tag{19}$$

If the map in Eq. (18) is unital then we can write the second term in the right-hand side as

$$\frac{1}{d} \sum_{i=1}^{r} \sum_{j,l,m=1}^{d^2} a_{ij} a^*_{il} S_{mln} d\delta_{jm} = \sum_{i=1}^{r} \sum_{j,l=1}^{d^2} a_{ij} a^*_{il} S_{jln} = \frac{1}{d}. \tag{20}$$

where we used $\sum_k S_{jkm} = d\delta_{jm}$ and Eq. (9). This says that if the map is unital then the map applies to the probability vector $\vec{p}$ directly:

$$p_n \mapsto p'_n = \sum_{i=1}^{r} \sum_{j,k,l,m=1}^{d^2} a_{ij} p_k a^*_{il} S_{jkm} S_{mln}. \tag{21}$$

Of course, Eq. (9) will not be true for non-unital maps so generally we would get an affine map

$$\vec{p} \mapsto \vec{p}' = M\vec{p} + \vec{t} \tag{22}$$

where

$$M_{jk} = \sum_{i=1}^{r} \sum_{l,m,n=1}^{d^2} a_{im} a^*_{in} S_{mkl} S_{lnj}, \tag{23}$$

$$t_j = \frac{1}{d(d+1)} \left(1 - \sum_{k=1}^{d^2} M_{jk} \right). \tag{24}$$

The following are some examples of simple maps for density operators:

1. Trace-preserving inversion map:

$$\rho \mapsto \frac{1}{d-1} \left(I - \rho \right). \tag{25}$$

For pure states, this maps $|\psi\rangle$ to some state orthogonal to $|\psi\rangle$. We have

$$\begin{aligned} \rho \mapsto & \frac{1}{d-1} \left[I - (d+1) \sum_i p_i \Pi_i + I \right] \\ &= \sum_i \left[\frac{2}{d(d-1)} - \frac{d+1}{d-1} p_i \right] \Pi_i \\ &\equiv \sum_i \left[(d+1) p'_i - \frac{1}{d} \right] \Pi_i, \end{aligned} \tag{26}$$

which implies that

$$\boxed{p_i' = \frac{1}{d-1}\left(\frac{1}{d} - p_i\right).} \tag{27}$$

For instance, if we consider the SIC basis state

$$\vec{p} = \left(\frac{1}{d}, \frac{1}{d(d+1)}, \ldots, \frac{1}{d(d+1)}\right)^T, \tag{28}$$

that is, the probability vector for $\rho = \Pi_1$, then

$$\vec{p}' = \left(0, \frac{1}{d^2-1}, \ldots, \frac{1}{d^2-1)}\right)^T. \tag{29}$$

This means that

$$\vec{p} \cdot \vec{p}' = \frac{1}{d(d+1)} \tag{30}$$

which is indeed the value of the scalar product for orthogonal pure states.

2. Completely positive inversion map (approximate universal NOT gate):

$$\rho \mapsto \frac{1}{d^2-1}\left(dI - \rho\right). \tag{31}$$

We have

$$\begin{aligned} \rho \mapsto & \frac{1}{d^2-1}\left[dI + I - (d+1)\sum_i p_i \Pi_i\right] \\ &= \sum_i \left[\frac{1}{d(d-1)} - \frac{p_i}{d-1}\right]\Pi_i \\ &\equiv \sum_i \left[(d+1)p_i' - \frac{1}{d}\right]\Pi_i, \end{aligned} \tag{32}$$

which implies that

$$\boxed{p_i' = \frac{1}{d^2-1}\left(1 - p_i\right).} \tag{33}$$

For instance, if we consider the SIC basis state

$$\vec{p} = \left(\frac{1}{d}, \frac{1}{d(d+1)}, \ldots, \frac{1}{d(d+1)}\right)^T \tag{34}$$

then

$$\vec{p}' = \left(\frac{1}{d(d+1)}, \frac{d^2+d-1}{d(d-1)(d+1)^2}, \ldots, \frac{d^2+d-1}{d(d-1)(d+1)^2}\right)^T. \tag{35}$$

3. Trace-preserving projection onto a random (average) pure state:

$$\rho \mapsto \frac{1}{d+1}\left(I + \rho\right). \tag{36}$$

We have

$$\begin{aligned}\rho \mapsto & \frac{1}{d+1}\left[I + (d+1)\sum_i p_i \Pi_i - I\right] \\ &= \sum_i p_i \Pi_i \\ &\equiv \sum_i \left[(d+1)p_i' - \frac{1}{d}\right]\Pi_i,\end{aligned} \tag{37}$$

which implies that

$$\boxed{p_i' = \frac{1}{d+1}\left(p_i + \frac{1}{d}\right).} \tag{38}$$

For instance, if we consider the SIC basis state

$$\vec{p} = \left(\frac{1}{d}, \frac{1}{d(d+1)}, \ldots, \frac{1}{d(d+1)}\right)^T \tag{39}$$

then

$$\vec{p}' = \left(\frac{2}{d(d+1)}, \frac{d+2}{d(d+1)^2}, \ldots, \frac{d+2}{d(d+1)^2}\right)^T. \tag{40}$$

4. Qudit depolarizing channel:

$$\rho \mapsto (1-\epsilon)\rho + \frac{\epsilon}{d}I. \tag{41}$$

We have

$$\begin{aligned}\rho \mapsto & (1-\epsilon)\left[(d+1)\sum_i p_i \Pi_i - I\right] + \frac{\epsilon}{d}I \\ &= (1-\epsilon)\left[(d+1)\sum_i p_i \Pi_i\right] - \left(1 - \epsilon - \frac{\epsilon}{d}\right)I \\ &= \left[(d+1)p_i(1-\epsilon) - \frac{1}{d}\left(1 - \frac{(d+1)\epsilon}{d}\right)\right]\Pi_i \\ &\equiv \sum_i \left[(d+1)p_i' - \frac{1}{d}\right]\Pi_i,\end{aligned} \tag{42}$$

which implies that

$$\boxed{p_i' = (1-\epsilon)p_i + \frac{\epsilon}{d^2}.} \tag{43}$$

For instance, if we consider the SIC basis state

$$\vec{p} = \left(\frac{1}{d}, \frac{1}{d(d+1)}, \ldots, \frac{1}{d(d+1)}\right)^T \tag{44}$$

then

$$\vec{p}' = \left(\frac{d(1-\epsilon)+\epsilon}{d^2}, \frac{d+\epsilon}{d^2(d+1)}, \ldots, \frac{d+\epsilon}{d^2(d+1)}\right)^T. \tag{45}$$

5. Qudit flip channel:

$$\rho \mapsto \sum_{i=0}^{d-1} \epsilon_i X^i \rho X^{i\dagger} \tag{46}$$

where ϵ_i represents the probability of $|j\rangle \mapsto |j \oplus i\rangle$ for the orthonormal basis $\{|j\rangle\}_{j=0}^{d-1}$ that is shifted by X,

$$X|j\rangle = |j \oplus 1\rangle \tag{47}$$

with $i \oplus j = i + j \mod d$ and later, $i \ominus j = i - j \mod d$. In this particular example, it is convenient to restrict ourselves to Weyl-Heisenberg SICs,

$$|\psi_{jk}\rangle = X^j Z^k |\psi\rangle, \qquad \Pi_{jk} = |\psi_{jk}\rangle\langle\psi_{jk}| \tag{48}$$

so the relation between density operators and probability vectors can be written as

$$\rho = \sum_{j,k=0}^{d-1} (d+1) p_{jk} \Pi_{jk} - I. \tag{49}$$

Thus,

$$\begin{aligned} \rho \mapsto & \sum_{i=0}^{d-1} \sum_{j,k=0}^{d-1} \epsilon_i \left[(d+1)p_{jk}\right] X^i \Pi_{jk} X^{i\dagger} - \sum_{i=0}^{d-1} \epsilon_i X^i X^{i\dagger} \\ &= (d+1) \sum_{i=0}^{d-1} \sum_{j,k=0}^{d-1} \epsilon_i p_{jk} \Pi_{(j\oplus i)k} - I \\ &= (d+1) \sum_{i=0}^{d-1} \sum_{j,k=0}^{d-1} \epsilon_i p_{(j\ominus i)k} \Pi_{jk} - I, \end{aligned} \tag{50}$$

which implies that

$$\boxed{p'_{jk} = \sum_{i=0}^{d-1} \epsilon_i p_{(j\ominus i)k}.} \tag{51}$$

For instance, if $\epsilon_i = \frac{1}{d}$ for all i and we consider the SIC basis state

$$p_{jk} = \frac{d\delta_{j0}\delta_{k0} + 1}{d(d+1)} \tag{52}$$

then

$$p'_{jk} = \begin{cases} \dfrac{2}{d(d+1)} & \text{if } k = 0, \\ \dfrac{1}{d(d+1)} & \text{if } k \neq 0. \end{cases} \tag{53}$$

6. Qutrit phase damping channel:

$$\rho \mapsto (1-\epsilon)\rho + \epsilon Z \rho Z^\dagger \tag{54}$$

where

$$Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \tag{55}$$

and $\omega = e^{i\frac{2\pi}{3}}$. Here we consider Weyl-Heisenberg SICs generated from the vectors

$$|\psi_{jk}\rangle = X^i Z^j |\psi\rangle \tag{56}$$

where

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{57}$$

and $|\psi\rangle$ is some fiducial vector. Writing the density operator as

$$\rho = \sum_{j,k=0}^{2} (d+1) p_{jk} \Pi_{jk} - I \tag{58}$$

and

$$\vec{p} = \left(p_{00}, p_{01}, p_{02}, p_{10}, p_{11}, p_{12}, p_{20}, p_{21}, p_{22}\right)^T, \tag{59}$$

it is straightforward to show that $\vec{p} \mapsto M\vec{p}$ where

$$M = \begin{pmatrix} B & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B \end{pmatrix}, \qquad B = \begin{pmatrix} 1-\epsilon & 0 & \epsilon \\ \epsilon & 1-\epsilon & 0 \\ 0 & \epsilon & 1-\epsilon \end{pmatrix}. \tag{60}$$

7. Qutrit amplitude damping channel, with Kraus operators

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{1-\epsilon} & 0 \\ 0 & 0 & \sqrt{1-\epsilon} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \sqrt{\epsilon} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & \sqrt{\epsilon} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{61}$$

where $\epsilon = 1 - e^{-\Gamma t}$ represents the decoherence parameter. We can compute the affine map directly from these matrices A_i if we express them as

$$A_i = \sum_{j,k=1}^{3} A_{ijk} E_{jk} \tag{62}$$

where E_{jk} are just the standard basis matrices

$$E_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \cdots \quad E_{33} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{63}$$

We can then compute a_{im} in Eq. (23) using

$$a_{il} = \sum_{j,k} A_{ijk} \mathrm{Tr}\left(E_{jk} D_l\right). \tag{64}$$

where D_l is the dual operator to the SIC projector Π_l

$$D_l = \frac{1}{3}\left(4\Pi_l - I\right). \tag{65}$$

We can choose any qutrit SIC for computing the structure coefficients, which might be tedious but is straightforward. We find that the affine map $\vec{p} \mapsto M\vec{p} + \vec{t}$ for the amplitude damping noise is given by

$$M = \frac{1}{3}\begin{pmatrix} B_1 & -B_2 & -B_2 \\ B_2 & B_3 & B_2 \\ B_2 & B_2 & B_3 \end{pmatrix} \tag{66}$$

where

$$B_1 = \begin{pmatrix} 3-4\epsilon & -\epsilon & -\epsilon \\ -\epsilon & 3-4\epsilon & -\epsilon \\ -\epsilon & -\epsilon & 3-4\epsilon \end{pmatrix}, \quad B_2 = \begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon \end{pmatrix}, \quad B_3 = \begin{pmatrix} 1+2\epsilon' & 1-\epsilon' & 1-\epsilon' \\ 1-\epsilon' & 1+2\epsilon' & 1-\epsilon' \\ 1-\epsilon' & 1-\epsilon' & 1+2\epsilon' \end{pmatrix}, \tag{67}$$

with $\epsilon' = \sqrt{1-\epsilon}$, and

$$\vec{t} = \frac{1}{6}\begin{pmatrix} 2\vec{v} \\ -\vec{v} \\ -\vec{v} \end{pmatrix}, \qquad \vec{v} = \begin{pmatrix} \epsilon \\ \epsilon \\ \epsilon \end{pmatrix}. \tag{68}$$