Implementing SIC-POVMs on a path-encoded optical storage loop

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In this note, I describe a theoretical model for how a SIC-POVM might be realized with an optical storage loop. This idea is based on the qutrit SIC-POVM experiment described in [1]. In that experiment, the quantum state is encoded in the path and polarization modes. Here I find it more convenient to describe a slightly different experiment that uses path-encoded qudits.

Consider some d-dimensional Weyl-Heisenberg SIC-POM with fiducial vector $|\phi\rangle$. Let $D_{ij} = \tau^{ij} X^i Z^j$ be the displacement operators for $i, j \in \{0, 1, \ldots, d-1\}$, where $\tau = -e^{i\pi/d}$, and X and Z are the d-dimensional shift and phase operators, respectively.

Let $|\psi_0\rangle \in \mathbb{C}^d$ be a path-encoded qudit that we want to measure. Define the $2d$-dimensional unitary $U_\epsilon = B_\epsilon \otimes I_d$, where

$$B_\epsilon = \begin{pmatrix} \sqrt{1-\epsilon} & \sqrt{\epsilon} \\ \sqrt{\epsilon} & -\sqrt{1-\epsilon} \end{pmatrix}. \tag{1}$$

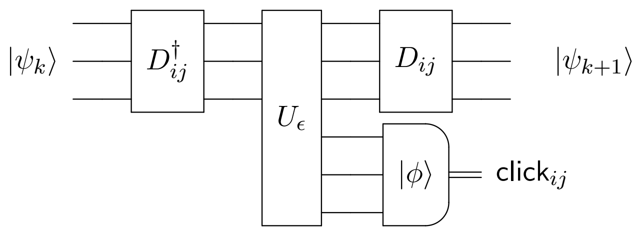
Observe that U_ϵ describes beam splitters being applied between modes n and $n+d$, for $n = 1, \ldots, d$. Here I make a simplifying assumption that the reflectivities for each pair of modes are the same.

For example, if we use $|\psi_0\rangle$ as an input for the first d modes of U_ϵ,

$$U_\epsilon \begin{pmatrix} |\psi_0\rangle \\ 0^d \end{pmatrix} = \begin{pmatrix} \sqrt{1-\epsilon}|\psi_0\rangle \\ \sqrt{\epsilon}|\psi_0\rangle \end{pmatrix} \tag{2}$$

so in some very rough sense, U_ϵ creates 2 copies of $|\psi_0\rangle$ on independent sets of modes but with different norms.

A quantum circuit for a "repeating" component of the storage loop model is shown below:



The idea is to have d^2 such components, one for each value of (ij), arranged sequentially in a loop. Let $|\psi_k\rangle$ denote the state after the signal has encounted U_ϵ exactly k times. The box labeled $|\phi\rangle$ is a binary measurement with projections $|\phi\rangle\langle\phi|$, which corresponds to getting outcome $\mathsf{click}_{ij} = 1$, and $I_d - |\phi\rangle\langle\phi|$, which corresponds to getting outcome $\mathsf{click}_{ij} = 0$. The output state $|\psi_{k+1}\rangle$ becomes the input for the next component.

The idea for doing the measurement is to send a photon into the storage loop and trapping it there. If ϵ is small, just a small part of the state gets leaked onto the last d modes of $U\epsilon$ and so only weak projections are performed on the initial state. Moreover, only the last d modes are measured so the amplitudes on the first d modes remain the same whether or not a click is registered in the measured modes.

Now I will compute outcome probabilities for the weak projections and show how they correspond to the outcome probabilities expected for a SIC-POVM. Let us look at the component (ij) again. Suppose the input state is $|\psi_k\rangle$. Since

$$U_\epsilon \begin{pmatrix} D^\dagger_{ij}|\psi_k\rangle \\ 0^d \end{pmatrix} = \begin{pmatrix} \sqrt{1-\epsilon} D^\dagger_{ij}|\psi_k\rangle \\ \sqrt{\epsilon} D^\dagger_{ij}|\psi_k\rangle \end{pmatrix} \tag{3}$$

the probability for getting a click when I measure $|\phi\rangle$ at component (ij) is

$$\Pr\left[\mathsf{click}_{ij} = 1 \,|\, |\psi_k\rangle\right] = \epsilon \left|\langle\phi|D^\dagger_{ij}|\psi_k\rangle\right|^2 = \epsilon\, |\langle\psi_k|\phi_{ij}\rangle|^2 , \tag{4}$$

where $|\phi_{ij}\rangle = D_{ij}|\phi\rangle$. This shows that the clicks are indeed related to projections onto the elements of a SIC-POVM. From Eq. (2), I have $\||\psi_k\rangle\| = \left(\sqrt{1-\epsilon}\right)^k$ so

$$\Pr\left[\mathsf{click}_{ij} = 1 \middle| |\psi_k\rangle\right] = \epsilon\left(\sqrt{1-\epsilon}\right)^k |\langle\psi_0|\phi_{ij}\rangle|^2, \tag{5}$$

Now in order to get the total probability, we need to add up contributions for all the times the signal passes through component (ij). If there are d^2 components in the loop, then we want to sum for values of k in steps of d^2. Fortunately, this calculation has a simple result when $\epsilon \to 0$:

$$\lim_{\epsilon\to 0} \sum_{m=0}^{\infty} \sum_{k=di+j+md^2} \epsilon(\sqrt{1-\epsilon})^k = \frac{2}{d^2} \tag{6}$$

Therefore,

$$p_{ij} = \Pr\left[\mathsf{click}_{ij} = 1\right] = \frac{2}{d^2} |\langle\psi_0|\phi_{ij}\rangle|^2, \tag{7}$$

Note that if we compute the probability of getting a click,

$$P = \Pr[\mathsf{click}] = \sum_{i,j=0}^{d-1} p_{ij} = \frac{2}{d^2}\langle\psi_0|\left(\sum_{i,j}|\phi_{ij}\rangle\langle\phi_{ij}|\right)|\psi_0\rangle = \frac{2}{d} \tag{8}$$

since the sum of those SIC projections is d times the identity operator. Finally, we obtain the SIC-POVM probabilities q_{ij} by normalizing, i.e., $q_{ij} = p_{ij}/P$.

It is worth noting that the model we have described gets a bit inefficient since the probability of getting a click for any particular photon varies inversely with d. Furthermore, observe that the experiment setup would also work for any symmetric quantum measurement that correspond to geometrically uniform states $|\psi_j\rangle = U_j|\psi\rangle$, that is, a measurement with POVM elements

$$\Pi_j = \frac{1}{n}\rho^{-1/2}|\psi_j\rangle\langle\psi_j|\rho^{-1/2}, \quad \rho = \frac{1}{n}\sum_j |\psi_j\rangle\langle\psi_j| = \frac{1}{d}I. \tag{9}$$

References

[1] ZED Medendorp, FA Torres-Ruiz, LK Shalm, GNM Tabia, CA Fuchs, and AM Steinberg. Experimental characterization of qutrits using symmetric informationally complete positive operator-valued measurements. *Physical Review A*, 83(5):051801, 2011.