Neutrino Mass and Neutrino Oscillations

Gelo Noel M. Tabia[1,*]

[1]*Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, Ontario N2L 2Y5, Canada*
Department of Physics and Astronomy and Institute for Quantum Computing, University of Waterloo,
200 University Avenue West, Waterloo, ON, N2L 3G1, Canada

Measurements performed in solar neutrino experiments in the 1960s and atmospheric neutrino experiments in the 1980s exhibited a deficit in the flux of neutrinos when compared to the predictions of the Standard Model. To explain this unexpected 'disappearance' of momentum and energy, neutrinos were hypothesized to transition between flavors in a mixing process known as neutrino oscillations, which was first proposed by Bruno Pontecorvo in 1957 as a distinct possibility for massive neutrinos. We discuss a few phenomenological models that fit known observational and experimental data and attempt to explain neutrino masses and flavor mixing. In particular, they generally provide bounds on the primary oscillation parameters—the mixing angles and the squared mass differences—in the context of various mass-generating mechanisms for neutrinos. In this project, we provide a basic overview of the theory of neutrino mass and neutrino oscillations. We also present some of the existing observational evidence from solar and atmospheric neutrinos as well as the experimental results from reactor and accelerator laboratory searches.

I. INTRODUCTION

In 1911, Lisa Meitner and Otto Hahn observed in beta decay experiments that the energies of the emitted electrons exhibited a continuous spectrum, confirmed in later experiments by Charles Drummond Ellis and colleagues in the 1920s. To reconcile these experimental observations with energy conservation, Wolfgang Pauli suggested a third unobserved particle in the decay process: a neutrino. For this purpose, the neutrino should carry no electric charge and have virtually no interactions with matter. Pauli himself pointed out that its mass should be no larger than 1 percent of the proton mass—the earliest known limit to neutrino mass. Thus, in standard radioactive decay, nuclei mutate to a different species when neutrons transform into protons according to

$$n \to p + e + \overline{\nu_e}. \tag{1}$$

The energy range for the emitted electron corresponded perfectly to the various ways energy can be distributed in a radioactive decay into a three-particle final state. In 1956, Fred Reines and Clyde Cowan Jr. first directly observed neutrinos produced by a nuclear reactor [1], a discovery that was awarded the Nobel Prize in 1995. The muon neutrino ν_μ was discovered in 1962 by Leon Lederman, Melvin Schwartz and Jack Steinberger (Nobel Prize, 1988) and the tau neutrino ν_τ almost forty years later in 2000 by the DONUT Collaboration at Fermilab in an experiment built specifically to detect it.

To explain Pauli's neutrino hypothesis, Enrico Fermi developed a theory of beta decay [2] in terms of the weakly interacting fields of a neutron, proton, electron and antineutrino. If we denote the fermion fields by ψ_j for $j = n, p, e, \bar{n}_e$ then Fermi's interaction Hamiltonian density can be written as

$$\mathcal{H}_{\text{weak}} = \frac{G_F}{\sqrt{2}}\overline{\psi_p}\gamma_\mu\psi_n\overline{\psi_e}\gamma^\mu\psi_{\nu_e} \tag{2}$$

where G_F is the Fermi coupling constant. The Hamiltonian in low-energy weak decays essentially retains the form as Fermi's interaction except for some modifications: (i) fundamental fields for proton and neutron are now in terms of quarks, (ii) before electroweak theory, Robert Marshak and George Sudarshan, and Richard Feynman and Murray Gell-Mann determined the correct V-A form for the currents, and (iii) weak interactions include both charged current ($W^\pm$) and neutral current (Z^0) interactions. Although Fermi's theory of weak interactions matches well with tree Feynman diagram calculations, it fails with loop calculations as the theory is not renormalizable in four dimensions–it requires modifications that were introduced later in electroweak theory, first discovered in the 1960 when Sheldon Glashow extended electroweak unification models due to Julian Schwinger by including a short range neutral current Z^0, and improved upon by Mohammad Abdus Salam and Steven Weinberg when they incorporated the Higgs mechanism.

For the most part, we know that the Standard Model provides a remarkable picture of the vast array of phenomena in particle physics, with the exception of gravitational effects. It is the most general theory consistent with general physical principles like Lorentz invariance and unitarity plus two assumptions: (i) renormalizability and (ii) observed particle content, all of which have been observed experimentally or has indirect experimental evidence (gluons, quarks) save for the yet-unseen Higgs boson.

There is however one phenomenon where it is clear that the Standard Model fails: explaining the phenomenon of neutrino oscillations. The Standard Model asserts separate conservation laws for the three lepton numbers L_e, L_μ and L_τ. Even taking into account electroweak

*Electronic address: gtabia@perimeterinstitute.ca

anomaly suggests quantities like $L_e - L_\mu$ or $L_\mu - L_\tau$ should be exactly conserved. Consequently, the theory predicts stable, massless neutrinos where charged current weak interactions only involve specific pairs (ν_ℓ, ℓ^-).

In 1957, Bruno Pontecorvo realized that non-zero neutrino masses imply the possibility of neutrinos oscillating from one flavor to another [3]. He proposed that the neutrino state produced in weak interactions is a superposition of states of two Majorana neutrinos with definite mass. Such a phenomenon is familiar in the quark sector of the Standard Model, where we have neutral kaon mixing. At the time, only the electron neutrino was known so this hypothesis of neutrino mixing was highly speculative.

The problem of searching for neutrino mass and studying the oscillation phenomenon has been faced in the past through many different experimental techniques. The first studies, based on the Fermi-Perrin method [4] of observing near the end point of the β-spectrum, obtained the limit $m_\nu = 500$ MeV [5], later improved in the 1950s to $m_\nu \lesssim 250$ MeV. Therefore, it became evident that if neutrinos had any mass at all, it would be lighter than an electron. After the discovery of parity violation in β-decay, proponents of the two-component spinor theory of neutrinos [6] suggested that the neutrino is a massless particle with a chiral field ν_L or ν_R. The experiment of Goldhaber et al. in 1958 [7] supported this hypothesis, establishing that neutrinos are left-handed particles.

At present, the effects of neutrino masses and mixing are investigated in various experimental setups. There are three kinds of experiments involving small neutrino masses: (i) neutrino oscillation experiments, (ii) search for neutrinoless double-beta decay, and (iii) measurements of electron neutrino mass in the high-energy β spectrum of ^3H. There are also those experimental results that favor massive neutrinos and neutrino mixing in (i) solar neutrinos (Homestake, Kamiokande, GALLEX, SAGE, Super-Kamiokande), (ii) atmospheric neutrinos (Super-Kamiokande, Kamiokande, IMB, Soudan, MACRO), and (iii) the accelerator LSND experiment.

In this project, we will provide a review of the basic theory and phenomenology of massive neutrinos and neutrino oscillations. In Section II, we will consider how categorize the place of neutrinos in the Standard Model and how extensions of it can accommodate neutrino masses. In Section III we describe the fundamental phenomenology of neutrino oscillations, where we show the relevant equations for explaining the experimental observations without going into any rigorous field theoretic derivation. In Section IV, we discuss the neutrino oscillation experiments and their key results. In particular, we are also interested in the upper and lower bounds for neutrino masses from direct searches and double beta decay experiments. In Section V, we discuss models that extend the Standard Model and generate neutrino masses. There is a significant amount of literature on the subject but in this report, our main references are [8–13].

II. NEUTRINO MASS AND THE STANDARD MODEL

One of the most beautiful aspects of modern theories of particles physics is the relation between forces mediated by spin-1 particles and local gauge symmetries. Within the Standard Model, the strong, weak and electromagnetic interactions are related to, respectively, $SU(3)$, $SU(2)$ and $U(1)$ gauge groups. Many features of the various interactions are then explained by the symmetry to which they are related. In particular, the way that the various fermions are affected by the different types of interactions is determined by their representations under the corresponding symmetry groups.

A. Basic neutrino properties

Neutrinos are fermions that have neither strong nor electromagnetic interactions. In group theory language, they are singlets of $SU(3)_{\mathrm{C}} \times U(1)_{\mathrm{EM}}$. Active neutrinos have weak interactions, that is, they are not singlets of $SU(2)_{\mathrm{L}}$. Sterile neutrinos, if they exist, have none of the Standard Model gauge interactions and they are singlets of the associated gauge group. It has half-integer spin with a very tiny mass as implied by the phenomena of neutrino oscillations. The existence of a neutrino mass also strongly suggests the existence of a tiny neutrino magnetic moment of the order of $10^{-19}\mu_B$, which might allow for small electromagnetic interactions. Neutrinos have been shown experimentally to always have left-handed chirality.

The Standard Model has three active neutrinos. They reside in lepton doublets,

$$L_\ell = \begin{pmatrix} \nu_{L,\ell} \\ \ell^-_L \end{pmatrix} \qquad \text{where } \ell = e, \mu, \tau. \tag{3}$$

Here e,μ and τ are the charged lepton mass eigenstates. The three neutrino interaction eigenstates ν_e,ν_μ and ν_τ are defined as the $SU(2)_{\mathrm{L}}$ partners of the mass eigenstates. The charged current interaction for leptons goes like

$$-\mathcal{L}_C = \frac{g}{2\sqrt{2}} \sum_\ell \left(\overline{\nu_{L\ell}} \gamma^\mu \ell^-_L W^+_\mu + \ell^+_L \gamma^\mu \nu_{L\ell} W^-_\mu \right) \tag{4}$$

Additionally, there are neutral current interactions for neutrinos of the form

$$-\mathcal{L}_{\mathrm{NC}} = \frac{g}{2\cos\theta_W} \sum_\ell \overline{\nu_{L\ell}} \gamma^\mu \nu_{L\ell} Z^0_\mu. \tag{5}$$

The charged and neutral current interactions yield all neutrino interactions in the Standard Model.

The measurement of the decay width of the Z^0 boson into neutrinos makes the existence of exactly three light ($m_\nu \lesssim m_Z/2$) active neutrinos an experimental fact. Expressed in terms of the Standard Model prediction for single neutrino generation, the observed data [14] implies that

$$\begin{aligned} N_\nu &= 2.994 \pm 0.012 \qquad \text{(Fit to LEP data)} \\ N_\nu &= 3.00 \pm 0.06 \qquad \text{(Invisible } Z \text{ width)} \end{aligned} \tag{6}$$

B. The Standard Model predicts $m_\nu = 0$

The Standard Model is based on the gauge group

$$G_{\rm SM} = SU(3)_{\rm C} \times SU(2)_{\rm L} \times U(1)_{\rm Y} \tag{7}$$

with three fermion generations, each generation consisting of 5 different representations of the gauge group:

$$\begin{aligned} Q_L\,(3,2)_{\frac{1}{6}}\,,\ U_R\,(3,1)_{\frac{2}{3}}\,,\ D_R\,(3,1)_{-\frac{1}{3}}\,, \\ L_L\,(1,2)_{-\frac{1}{2}}\,,\ E_R\,(1,1)_{-1}\,. \end{aligned} \tag{8}$$

The notation above reads in the following way: for example, a left-handed lepton field L_L is a singlet of the $SU(3)_{\rm C}$ group, a doublet (2) of the $SU(2)_{\rm L}$ group, and carries hypercharge -1/2 under the $U(1)_{\rm Y}$ group.

The electroweak gauge symmetry is broken by the non-vanishing vacuum expectation value of the Higgs field, through the Higgs mechanism. Using a doublet Higgs representation ϕ

$$\phi = \begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix}, \tag{9}$$

with Higgs potential

$$V(\phi) = -\mu\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2 \tag{10}$$

then for $\mu^2 > 0$, the vacuum expectation

$$\langle\phi_0\rangle = \sqrt{\frac{\mu^2}{2\lambda}} \equiv \frac{v}{\sqrt{2}} \tag{11}$$

gives mass to the charged bosons and Z^0

$$M_W = \frac{1}{2}gv, \qquad \cos\theta_W = \frac{M_W}{M_Z} \tag{12}$$

where θ_W is known as the Weinberg angle for neutral gauge bosons. The gauge boson orthogonal to Z^0 is the massless photon. Every particle that couples to the Higgs field acquires a mass.

Chiral symmetry forbids a bare mass term for the fermions; fermion masses arise from the Yukawa interactions,

$$\begin{aligned} -\mathcal{L}_{\rm Yukawa} &= Y^u_{ij}\overline{Q_{Li}}i\tau_2\phi^* U_{Rj} + Y^d_{ij}\overline{Q_{Li}}\phi D_{Rj} \\ &\quad + Y^\ell_{ij}\overline{L_{Li}}\phi E_{Rj} + \text{(h.c.)} \end{aligned} \tag{13}$$

after spontaneous symmetry breaking. The Yukawa interactions give masses to charged fermions but neutrinos remain massless. In general, neutrinos can have a mass term of the Majorana type, which requires just one helicity state; however, since the ν_L is part of the $SU(2)_{\rm L}$ doublet field and possesses lepton number +1, the Majorana mass term $\nu_L^T C^{-1}\nu_L$, where C is the Lorentz conjugation matrix, transforms as an $SU(2)_{\rm L}$ triplet—it is not gauge invariant. It also violates the lepton number symmetry by 2 units.

Initially, we might think it is possible to induce the neutrino mass from non-perturbative effects. In the Standard Model, the only known source of lepton number violation involves the so-called weak instanton effects. However, even though the lepton number current conservation is indeed broken non-perturbatively though a chiral anomaly, it turns out an identical contribution can be found for the baryon number current in such a way that the $B-L$ current is conserved to all orders in the gauge coupling. Because the neutrino mass operator above also violates $B-L$, this shows that neutrino mass vanishes even in the presence of non-perturbative corrections. Consequently, we expect neither mixing nor CP violation in the leptonic sector.

C. Extensions allow $m_\nu \neq 0$

There are many good reasons to think that the Standard Model provides an inadequate description of all known physical phenomena: the problem of a fine-tuned Higgs mass might be solved by supersymmetry; gauge coupling unification and the variety of gauge representations may be explained well by grand unified theories; and the existence of gravity suggests extensions like string theories are relevant to Nature. If any of the proposed modifications is realized in Nature, the Standard Model becomes an effective field theory, valid as a low-energy approximation to a more general theory up to some energy scale $\Lambda_{\rm NP}$ characterizing new physics.

Even as an effective low energy theory, the Standard Model would still retain the gauge group, the spectrum of fermions, and the pattern of spontaneous symmetry breaking as valid ingredients to describe Nature at energies much smaller than $\Lambda_{\rm NP}$. The Standard Model predictions, however, are modified with corrections proportional to powers of $E/\Lambda_{\rm NP}$. This means that a more general field theory than the Standard Model would have to incorporate non-renormalizable terms.

A more general theory of new physics won't have to respect the accidental symmetries of the Standard Model. Indeed it is known that there is a set of dimension-5 terms made of Standard Model fields consistent with gauge symmetry but violates

$$G_{\rm global} = U(1)_{\rm B} \times U(1)_{\rm e} \times U(1)_\mu \times U(1)_\tau, \tag{14}$$

the accidental global symmetry corresponding to baryon number and lepton flavor numbers. These terms are of the form

$$\frac{Z^\nu_{ij}}{\Lambda_{\rm NP}}\phi\phi L_{Li}L_{Lj}. \tag{15}$$

In particular, eq. (15) violates L by two units and leads to neutrino masses:

$$(M_\nu)_{ij} = \frac{Z^\nu_{ij}}{\Lambda_{\rm NP}} \frac{v^2}{2}. \tag{16}$$

This provides a generic extension of the Standard Model which, among other things, would natural explain why the neutrino mass is small (it scales with the inverse of the large energy scale of new physics), and how the lepton flavor symmetries are broken, allowing for lepton mixing and CP violations.

The best known scenario that leads to eq. (16) is the *see-saw mechanism* [27], where we assume the existence of heavy sterile neutrinos with bare mass terms and Yukawa interactions that result in a mass matrix

$$M_\nu = \begin{pmatrix} 0 & Y^\nu \frac{v}{\sqrt{2}} \\ (Y^\nu)^T \frac{v}{\sqrt{2}} & M_N \end{pmatrix} \tag{17}$$

Whenever the eigenvalues of M_N are much larger than the electroweak breaking scale v, diagonalization of M_ν leads to three light mass eigenstates with a mass matrix (16). We will see more details of ways to go beyond the Standard Model in Section V.

III. THEORY OF NEUTRINO OSCILLATIONS

A. Dirac and Majorana fermions

Massive fermions can either be of the Dirac or Majorana type. If they carry electric charge, then they are necessarily Dirac fermions. Electrically neutral particles such as neutrinos are expected to by Majorana fermions on rather general grounds, no matter how they acquire their mass. Phenomenological differences between Dirac and Majorana neutrinos are tiny for most purposes because neutrinos are very light and the chiral weak interactions are well described in the V-A form. This knowledge is basic material, however, and it is useful to briefly review them here.

Consider the free Dirac field, whose field operator is given by

$$\psi = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2p^0}} \sum_s \left[u_s e^{-ip\cdot x} a_s + v_s e^{ip\cdot x} a_s^\dagger\right] \tag{18}$$

where $s = \pm 1/2$, $a_s^\dagger(p), a_s(p)$ are creation and annihilation operators, and the coefficients $u_s(p), v_s(p)$ satisfy

$$(\gamma^\mu p_\mu - m) u_s(p) = 0, \quad (\gamma^\mu p_\mu + m) v_s(p) = 0 \tag{19}$$

with γ^μ being the usual Dirac gamma matrices that obey the anti-commutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{20}$$

Since the particle and antiparticle are identical for a Majorana neutrino, it feels like $\psi(x)$ in that situation would be directly related to $\psi(x)^*$. However, if we just naively impose $\psi = \psi^*$ we eventually run into terms that are not Lorentz covariant. Instead we define a conjugate field

$$\hat{\psi(x)} \equiv \gamma_0 C \psi^*(x) \tag{21}$$

where the specific form of the conjugation matrix $C = i\gamma_2\gamma_0$ depends on the representation used for γ^μ.

The condition for the Majorana field in terms of the Dirac field is then obtained by setting

$$\psi(x) = \hat{\psi}(x). \tag{22}$$

If we express the Dirac bispinor as

$$\psi_D(x) = \begin{pmatrix} \chi(x) \\ \sigma^2 \phi^*(x) \end{pmatrix} \tag{23}$$

where $\chi(x)$ is a left-handed Weyl spinor then the Majorana spinor is obtained in the case when $\phi = \chi$. In this way, the Dirac fermion is shown to be equivalent to two Majorana fermions of equal mass.

If we introduce a two-component spinor ρ such that

$$\chi = \frac{1}{\sqrt{2}}(\rho_2 + i\rho_1), \qquad \phi = \frac{1}{\sqrt{2}}(\rho_2 - i\rho_1), \tag{24}$$

then the $U(1)$ symmetry of the Dirac Lagrangian

$$\mathcal{L}_D = \overline{\psi}\gamma^\mu \partial_\mu \psi - m\overline{\psi}\psi, \tag{25}$$

which for ψ given by eq. (23) yields

$$\mathcal{L}_D = -i \sum_\alpha \rho_\alpha{}^\dagger \sigma^\mu \partial_\mu \rho_\alpha - \frac{m}{2} \sum_\alpha \rho_\alpha^T \sigma^2 \rho_\alpha + (\text{h.c.}), \tag{26}$$

corresponds to a continuous rotation symmetry between components of ρ, that is,

$$\psi'_D = e^{i\alpha}\psi_D \implies \begin{pmatrix} \rho'_1 \\ \rho'_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} \tag{27}$$

which result from the mass degeneracy between ρ_1, ρ_2, indicating that the concept of fermion number is not basic.

The mass term in eq. (26) vanishes unless ρ and ρ^* anti-commute, so the quantized field of interest is the Majorana fermion at the onset. The solutions to eq. (26) is identical in form to the Dirac field in eq. (18)

$$\psi_M = \int \frac{d^3p}{(\sqrt{2\pi})^3} \frac{1}{\sqrt{2E}} \sum_s \left[u_s e^{-ip\cdot x} a_s + v_s e^{ip\cdot x} a_s^\dagger\right] \tag{28}$$

but with $u = C\overline{v}^T$. The expression in eq. (28) differs from the usual Fourier expansion for the Dirac spinor in eq. (18) in two significant ways: (i) the spinor is two-component, as there is a chiral projection acting of $u_s(p)$ and $v_s(p)$ and (ii) there is only one Fock space since particle and anti-particle coincide.

For completeness, it might be worth noting that two independent propagators follow from eq. (26):

$$\begin{aligned}\langle 0|\,\rho(x)\rho^*(y)\,|0\rangle &= i\sigma^\mu\partial_\mu\Delta_F(x-y),\\ \langle 0|\,\rho(x)\rho(y)\,|0\rangle &= m\sigma^2\partial_\mu\Delta_F(x-y),\end{aligned} \tag{29}$$

where $\Delta_F(x-y)$ is the usual Feynman propagator. The first one is the "normal" propagator that intervenes in total lepton-number-conserving ($\Delta L = 0$) processes, while the other one describes the virtual propagation of Majorana neutrinos in $\Delta L = 2$ processes such as neutrino-less double beta decay.

B. The PMNS mixing matrix

To include neutrino mixing in the Standard Model, one should first decide whether new neutrino flavors need to be introduced. For now we suppose that there are no more neutrino states beyond those in the Standard Model. We can then give a phenomenological description of how neutrinos convert from one species to another by adding an explicit mass term in the Standard Model Lagrangian

$$\mathcal{L}_{M_\nu} = \frac{1}{2}(M_\nu)_{ij}(\overline{\nu_i}P_L\nu_j) + (\text{h.c.}) \tag{30}$$

for some arbitrary complex symmetric 3×3 matrix, where $P_L = \frac{1}{2}(1+\gamma_5)$. With massive neutrinos, neutrino oscillations follow when the different neutrino states are allowed to mix.

Consider a neutrino beam created in a charged current interaction along with an antilepton $\overline{\ell}$. We call the neutrino ν_ℓ. In general, this is not a physical particle but a superposition of physical fields ν_a with different masses m_a

$$|\nu_\ell\rangle = \sum_\alpha U_{\ell\alpha}\,|\nu_\alpha\rangle\,. \tag{31}$$

For simplicity, we assume that the 3-momentum of the beam are the same but because the masses are different, the energies of all components can't all be equal. In this case, after time t, the state evolves into

$$|\nu_\ell(t)\rangle = \sum_\alpha e^{-iE_\alpha t}U_{\ell\alpha}\,|\nu_\alpha\rangle \tag{32}$$

where we suppose for the moment that the neutrinos are stable.

Eq. (32) represents a different superposition of states compared to eq. (31). The probability amplitude of finding the original state in the new beam is then given by

$$\langle\nu_\ell|\nu_{\ell'}(t)\rangle = \sum_\alpha e^{-iE_\alpha t}U_{\ell'\alpha}U^*_{\ell\alpha} \tag{33}$$

since the mass eigenstates are orthonormal, i.e. $\langle\nu_\beta|\nu_\alpha\rangle = \delta_{\alpha\beta}$. Thus, at any time t, the probability of ν_ℓ surviving in the beam is

$$\begin{aligned}\Pr[\nu_\ell|\nu_{\ell'}(t)] &= |\,\langle\nu_\ell|\nu_{\ell'}(t)\rangle\,|^2\\ &= \sum_{\alpha,\beta}\left|U_{\ell\alpha}U^*_{\ell'\alpha}U^*_{\ell\beta}U_{\ell'\beta}\right|\\ &\quad\times\cos\left[(E_\alpha - E_\beta)t - \phi_{\ell\ell'\alpha\beta}\right]\end{aligned} \tag{34}$$

where $\phi_{\ell\ell'\alpha\beta} = \arg\left(U_{\ell\alpha}U^*_{\ell'\alpha}U^*_{\ell\beta}U_{\ell'\beta}\right)$.

In all situations of practical interest, neutrinos are highly relativistic, so we are allowed to make the approximation

$$E_\alpha = |\vec{p}| + \frac{m_\alpha^2}{2|\vec{p}|} \tag{35}$$

and the time t to be replaced by the distance x traveled by the beam. Thus,

$$\Pr[\nu_\ell|\nu'_\ell] = \sum_{\alpha\beta}\left|U_{\ell\alpha}U^*_{\ell'\alpha}U^*_{\ell\beta}U_{\ell'\beta}\right|\cos\left(\frac{2\pi x}{L_{\alpha\beta}} - \phi_{\ell\ell'\alpha\beta}\right) \tag{36}$$

where the oscillation lengths $L_{\alpha\beta}$,

$$L_{\alpha\beta} \equiv \frac{4\pi|\vec{p}|}{\Delta m^2_{ab}}, \qquad \left(\Delta m^2_{ab} = m_a^2 - m_b^2\right) \tag{37}$$

determine the relevant length scale for which oscillation effects stay appreciable.

Observe that if $x = \kappa L_{\alpha\beta}$ for some positive integer κ,

$$\Pr[\nu_\ell|\nu'_\ell] = \delta_{\ell\ell'}. \tag{38}$$

This means that the interesting non-trivial effects we want to observe experimentally happen in that regime in between multiples of $L_{\alpha\beta}$.

In the simplest case where only two flavors of neutrinos participate in oscillations, we have a rather simple mixing matrix

$$U = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix} \tag{39}$$

so the survival probability of ν_ℓ reduces to

$$\Pr[\nu_\ell|\nu'_\ell] = 1 - \sin^2 2\theta\sin^2\left(\frac{x\Delta m^2_{12}}{4|\vec{p}|}\right). \tag{40}$$

Of course, we know that there are (at least) three neutrinos in Nature so to be more realistic we should consider mixing across three generations of neutrinos. For simplicity, we start with the CP-preserving case where U is a real-valued matrix. Here the oscillation probabilities are

$$\begin{aligned}\Pr[\nu_\ell|\nu'_\ell] &= \sum_\alpha(U_{\ell\alpha}U_{\ell'\alpha})^2 +\\ &\quad 2\sum_{\alpha>\beta}U_{\ell\alpha}U_{\ell'\alpha}U_{\ell\beta}U_{\ell'\beta}\cos\left(\frac{x\Delta m^2_{\alpha\beta}}{2|\vec{p}|}\right)\\ &= \delta_{\ell\ell'} - 4\sum_{\alpha>\beta}U_{\ell\alpha}U_{\ell'\alpha}U_{\ell\beta}U_{\ell'\beta}\sin\left(\frac{x\Delta m^2_{\alpha\beta}}{4|\vec{p}|}\right).\end{aligned} \tag{41}$$

The most general mixing matrix excluding CP-violation effects is given by

$$U = \begin{pmatrix} c_{12}c_{13} & -s_{12}c_{13} & s_{13} \\ s_{12}c_{23} + c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & -s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13} & c_{12}s_{23} + s_{12}c_{23}s_{13} & c_{23}c_{13} \end{pmatrix} \quad (42)$$

where $c_{\alpha\beta} = \cos\theta_{\alpha\beta}$ and $s_{\alpha\beta} = \cos\theta_{\alpha\beta}$. This mixing matrix, which looks similar to the Cabibbo-Kobayashi-Maskawa matrix for quark-mixing weak decays, is called the PMNS matrix—after Pontecorvo, Maki, Nakagawa and Sakata—and it describes the amplitude in which a particular neutrino type ν'_ℓ participates in a charged-current interaction with a charged lepton ℓ. This matrix differs from the mass matrix in the Lagrangian because of the appearance of extra phases in the charged current mixing term which introduce CP-violation in neutrino interactions. These Majorana phases disappear in the expressions for probabilities, which is why the PMNS matrix is sufficient for neutrino oscillation observations even though it can not distinguish between the possibilities of Dirac and Majorana fermions.

It is interesting to look at a couple of limiting cases, the first one being

$$\frac{x\Delta m_{12}^2}{2|\vec{p}|} \ll 1 \quad (43)$$

which leads to

$$\Pr[\nu_\ell|\nu'_\ell] = \delta_{\ell\ell'} - 4U_{\ell\beta}U_{\ell'\beta} \times (\delta_{\ell\ell'} - U_{\ell 3}U_{\ell' 3})\sin^2\left(\frac{x\Delta m_{32}^2}{4|\vec{p}|}\right). \quad (44)$$

In this case, the survival probability $\ell' = \ell$ is just like in eq. (40).

The other limiting case of interest is

$$\frac{x\Delta m_{3j}^2}{2|\vec{p}|} \gg 1, \qquad (j = 1, 2) \quad (45)$$

where the oscillatory terms in the probability average out when we integrate over the energy spectrum of ν_ℓ, leading to

$$\begin{aligned} \Pr[\nu_\ell|\nu'_\ell(t)] &= \delta'_\ell\ell - 2U_{\ell 3}U_{\ell' 3}(\delta_{\ell\ell'} - U_{\ell 3}U_{\ell' 3}) \\ &\quad -4U_{\ell 1}U_{\ell' 1}U_{\ell 2}U_{\ell' 2}\sin^2\left(\frac{x\Delta m_{21}^2}{4|\vec{p}|}\right) \end{aligned} \quad (46)$$

If we had an electron neutrino, the survival probability would be given by

$$\begin{aligned} \Pr[\nu_e|\nu_e(t)] &= \cos^4\theta_{13}\left[1 - \sin^2 2\theta_{12}\sin^2\left(\frac{x\Delta m_{21}^2}{4|\vec{p}|}\right)\right] \\ &\quad +\sin^4\theta_{13} \end{aligned} \quad (47)$$

which for small θ_{13} reduces to eq. (40) with $\ell = \ell' = e$.

C. Oscillations with unstable neutrinos

The analysis above changes quite dramatically if neutrinos are unstable [15]. To see how the probabilities are modified, we consider the simple case of two-flavor mixing. Suppose the neutrino states are ν_e and ν_μ

$$|\nu_\ell\rangle = U_{\ell 1}|\nu_1\rangle + U_{\ell 2}|\nu_2\rangle, \qquad \ell = e, \mu \quad (48)$$

where ν_1, ν_2 are the mass eigenstates and U is the two-flavor mixing matrix (39) as before. Suppose that the mass $m_{\nu_1} > m_{\nu_2}$ then we can consider the decay process

$$\nu_1 \to \nu_2 + X, \quad \text{any } X \text{ with decay lifetime } \Gamma^{-1}. \quad (49)$$

Thus, suppose we have some initial state

$$|\nu_{\rm in}\rangle = \cos\alpha|\nu_e\rangle + \sin\alpha|\nu_\mu\rangle \quad (50)$$

then after time t this state is generally expected to evolve into

$$\begin{aligned} |\nu(t)\rangle &= e^{-iE_1t}e^{-\Gamma t/2}\sin(\theta+\alpha)|\nu_1\rangle \\ &\quad +e^{-iE_2t}\cos(\theta+\alpha)|\nu_2\rangle + \sum_k c_k|\nu_k, X\rangle \end{aligned} \quad (51)$$

where the last term corresponds to the final state of the ν_1-decay.

Hence, the survival probabilities for both the ν_e and ν_μ components of the initial beam are given by

$$\begin{aligned} \Pr[\nu_e|\nu_{\rm in}] &= \cos^2(\theta+\alpha)\cos^2\theta + e^{-\Gamma t}\sin^2(\theta+\alpha)\sin^2\theta \\ &\quad +\frac{1}{2}e^{-\Gamma t/2}\sin 2\theta\sin 2(\theta+\alpha)\cos\frac{t\Delta m^2}{2|\vec{p}|}, \\ \Pr[\nu_\mu|\nu_{\rm in}] &= \cos^2(\theta+\alpha)\sin^2\theta + e^{-\Gamma t}\sin^2(\theta+\alpha)\cos^2\theta \\ &\quad -\frac{1}{2}e^{-\Gamma t/2}\sin 2\theta\sin 2(\theta+\alpha)\cos\frac{t\Delta m^2}{2|\vec{p}|}. \end{aligned}$$

In the limit of fast decays, $\Gamma t \gg 1$, these expressions simplify into

$$\begin{aligned} \Pr[\nu_e|\nu_{\rm in}] &= \cos^2(\theta+\alpha)\cos^2\theta, \\ \Pr[\nu_\mu|\nu_{\rm in}] &= \cos^2(\theta+\alpha)\sin^2\theta, \end{aligned} \quad (52)$$

which can correspond to long distances traveled.

To see how this compares with the scenario with stable neutrinos, take the case of an initial beam of electron neutrinos with $\alpha = 0$. In the fast decay limit,

$$\Pr[\nu_e|\nu_e(t)] = \cos^4\theta, \quad (53)$$

which can become very small close to $\pi/2$. Compare this to eq. (40), which averaging over the distance x gives

$$\Pr[\nu_e|\nu_e(t)] = 1 - \frac{1}{2}\sin^2 2\theta > \frac{1}{2} \quad (54)$$

for all values of θ. Therefore, the physical impact of unstable neutrinos is rather significant.

D. Oscillations in matter and the MSW effect

So far our treatment of neutrino oscillations assumes that the neutrino propagates in vacuum. Because neutrinos hardly interact, this is a generally good approximation. To include medium effects, the standard approach is to consider changes in the interactions as a change in effective neutrino mass, similar to how the refractive index is modified for photons.

Consider neutrinos propagating in an environment like the interior of the sun. We can approximate this environment as a uniform background medium if its fluctuations are tiny enough. In this case, a mean field approximation is sufficient to describe the charged current interactions between say the electrons and neutrinos

$$\mathcal{L}_{C,\text{eff}} = \frac{G_F}{\sqrt{2}} \left[i\bar{e}\gamma^\mu (1+\gamma_5)\nu_e \right] \left[i\bar{\nu}\gamma_\mu (1+\gamma_5) e \right]. \quad (55)$$

Using a Fierz transformation, this can be rewritten as

$$\begin{aligned}\mathcal{L}_{C,\text{eff}} = \; & \frac{G_F}{\sqrt{2}} \; \left[i\overline{\nu_e}\gamma_\mu (1+\gamma_5)\nu_e \right] \left[i\bar{e}\gamma^\mu (1+\gamma_5) e \right] \\ & +(\text{other terms}).\end{aligned} \quad (56)$$

where we can easily read off the electron 4-current

$$J_e^\mu = i\bar{e}\gamma^\mu e. \quad (57)$$

There is also an axial current $i\bar{e}\gamma^\mu\gamma_5 e$ that vanishes in any parity-invariant environment like the Sun's interior.

The neutrino propagation term in the effective Lagrangian becomes

$$\mathcal{L}_\nu = \frac{G_F}{\sqrt{2}} n_e \left[i\overline{\nu_e}\gamma^0 (1+\gamma_5)\nu_e \right] \quad (58)$$

where n_e is the local electron number density. Note that $\mathcal{L}_\nu$ shifts the neutrino energies by $n_e G_F/\sqrt{2}$. None of the other charged-current interaction terms for neutrino vanish.

To see how the medium-dependent terms affect neutrino propagation, we can look at the two-flavor case. Since the new term merely shift the electron neutrino energy, the correction to electron neutrino Hamiltonian H_{ν_e} is

$$\delta H_{\nu_e} = \sqrt{2} G_F n_e \left| \nu_e \right\rangle \left\langle \nu_e \right|. \quad (59)$$

This is added to H_{ν_e} in vacuum, here understood to be in the flavor basis $\{\left|\nu_e\right\rangle, \left|\nu_\mu\right\rangle\}$:

$$H_{\nu_e} = E + \frac{m_1^2 + m_2^2}{4E} + \frac{\Delta m^2}{4E} \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \quad (60)$$

where $E \approx |\vec{p}|$ since neutrinos are highly relativistic in all practical situations.

The resulting effective mass matrix leads to

$$H_{\text{eff}} = \frac{n_e G_F}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\Delta m^2}{4E} \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \quad (61)$$

where terms proportional to the unit matrix are gone because they don't affect oscillation probabilities.

An important special case is when the medium-dependent term becomes maximal, i.e.,

$$\frac{n_e G_F}{\sqrt{2}} = \left(\frac{\Delta m^2}{4E} \right) \cos 2\theta \quad (62)$$

a condition that defines the resonance conditions, also called the *MSW effect*, after Mikheyev, Smirnov and Wolfenstein [16].

In essence, neutrino propagation in uniform matter is an exercise in degenerate perturbation theory: the degeneracy in the vacuum states of massless ν_e and ν_μ is broken by two small effects: the neutrino mass matrix and a medium-dependent term. Therefore, large mixings are possible even with small mixing angles provided that these two effects are roughly the same order of magnitude.

When the medium is non-uniform, the situation is not significantly different provided the following adiabaticity condition is fulfilled:

$$\frac{d\tilde{\theta}}{dx} \ll \left| \frac{\Delta m_{21}^2}{2E} \right| \quad (63)$$

where $\tilde{\theta}$ is the effective mixing angle.

For solar neutrinos, adiabatic evolution through resonance is a fair approximation of what happens during propagation. When these neutrinos hit earth, the total survival probability for ν_e goes like

$$\Pr[\nu_{e,\oplus} | \nu_{e,\odot}] = \sin^2\theta_p \sin^2\theta_d + \cos^2\theta_p \cos^2\theta_d \quad (64)$$

where θ_p is the medium-dependent mixing angle during solar production and θ_d is the vacuum mixing angle during detection. The numbers obtained from this analysis match reasonably well to experimental data to date.

E. Sterile neutrinos

Another possibility for explaining neutrino oscillations come from supplementing the Standard Model neutrinos with additional states. In general one might add N Majorana neutrino fields $s_x, x = 1, 2, ..., N$. Additional fields means more freedom in the kinds of interactions which can be entertained. We assume that the new fermions couple only through the mass matrix and so typically interact more weakly with matter than even the regular neutrinos—this lack of interaction with ordinary matter is why they are deemed sterile neutrinos.

Sterile neutrinos lead to new terms in the Standard Model Lagrangian of the form

$$\begin{aligned}\mathcal{L}_s \; = \; & -\tfrac{1}{2}\bar{s}_x\gamma^\mu\partial_\mu s_x - \tfrac{1}{2}\Big[M_{xy}\bar{s}_x P_L s_y \\ & + m_{ab}\overline{\nu}_a P_L \nu_b + 2\mu_{ax}\overline{\nu}_a P_L s_x + (\text{h.c.}) \Big]\end{aligned} \quad (65)$$

where the neutrino mass matrix

$$\begin{pmatrix} m & \mu \\ \mu^T & M \end{pmatrix} \tag{66}$$

is an arbitrary symmetric $(3+N)\times(3+N)$ matrix.

The freedom to choose μ, m, M leads to various mechanism for neutrino physics so we will only briefly summarize a few possibilities: If lepton number is conserved, then we have $N = 3$ with neutrino states such that

$$P_L \nu_a \to e^{i\omega} P_L \nu_a, \quad \text{and} \quad P_L s_a \to e^{-i\omega} P_L s_a. \tag{67}$$

In this case, provided $m = M = 0$, the mass term is invariant for any choice of μ. Then, it becomes convenient to group the 6 Majorana fields ν_a, s_a into 3 Dirac fields

$$\psi_a = \begin{pmatrix} \nu_a \\ s_a \end{pmatrix} \qquad \text{s. t. } \psi_a \to e^{i\omega}\psi_a. \tag{68}$$

We get the usual charged current mixing matrix

$$\mathcal{L}_C = \frac{ig}{\sqrt{2}} U_{ai} W_\mu \left(\bar{\ell}_a \gamma^\mu \gamma_L \psi_i\right) + (\text{h.c.}) \tag{69}$$

where U_{ai} is the usual PMNS matrix (42). This is identical to the scenario with 3 Majorana neutrinos except without the CP-violating phases. Furthermore, there are no off-diagonal couplings in the neutral-current interactions, which implies that the sterile neutrinos don't couple to Z^0.

Without the CP-violating phases, the Dirac neutrino model works precisely the same as the Majorana neutrinos because of helicity suppression: the ratio of amplitudes for detecting ν_a to s_a is typically of the order p^0/m, which in the lab-frame corresponds to the probability of ν_a detection at least 10^{12} times more likely than s_a scattering. It is this very small probability of getting a right-handed sterile neutrino that makes them practically irrelevant in the neutrino phenomenology.

More generic sterile neutrinos can be distinguished according to the relative sizes of μ, m, M:

1. $\mu \ll m, M$: Sterile neutrinos do not mix with the ordinary neutrinos in any significant way; therefore, for all practical purposes they don't couple at all with the usual Standard Model particles.

2. $\mu \gg m, M$: This case reduces to the Dirac neutrinos above when $m = M = 0$. If $N > 3$, this scenario leads to 3 Dirac neutrinos whose squared masses are the eigenvalues of $\mu\mu^\dagger$, plus $(N-3)$ massless sterile neutrinos which don't participate in weak interactions. If m, M are not exactly zero, then we have a perturbation of the Dirac case. The almost degenerate pairs of neutrinos are close in mass that should become particle-antiparticle pairs in the vanishing limit. For the same reason, the mixing angle of these nearly degenerate states is also almost maximal. From available data, such pseudo-Dirac neutrinos are ruled out for $m, M \gtrsim 10^{-9}$ eV.

Neutrino type	$\lvert U_{s\nu}\rvert \lesssim x$	$\Delta m^2_{14} \lesssim y$
Solar	0.001	10^{-4}
Reactor	0.1	10^{-3}
Atmospheric	0.2	10^{-3}
Supernova	0.01	10
Nucleosynthesis	0.1	10^{-8}
CMB	0.001	10

TABLE I. Currently known data imposing limits on existence of sterile neutrinos. $|U_{s\nu}|$ is the sterile-active mixing element and Δm^2_{14} is the relevant squared mass difference range in units of eV^2

3. $m \approx \mu \approx M$: When all the mass matrices are comparable, the observed data suggests light sterile neutrinos such that $m, \mu, M \approx 10^{-2}$ eV form which we can expect large oscillation effects among all neutrino eigenstates. Because of the lack of evidence for sterile neutrinos, this model is not favorable.

4. $m \ll \mu \ll M$: In this scenario, there are N heavy eigenstates with eigenvalues given by $\sqrt{M^\dagger M}$ as well as 3 mass eigenstates with eigenvalues from $\sqrt{\mathcal{M}^\dagger \mathcal{M}}$, with

$$\mathcal{M} = \mu M^{-1} \mu^T + m. \tag{70}$$

The heavy eigenstates have mixing angles with sterile neutrinos of the order $O(\mu/M)$. The light eigenstates are almost pure flavor eigenstates but also have order $O(\mu/M)$ sterile mixing. This way of obtaining neutrino masses is called the see-saw mechanism because increasing the mass from M decreases the mass from $\mathcal{M}$, and vice-versa.

Table (I) summarizes the constraints on the existence of sterile neutrinos based on different experiments and observations [9].

IV. EXPERIMENTAL STUDIES OF NEUTRINO MASS AND MIXING

Here we categorize the experiments aiming to measure the neutrino mass and to test neutrino oscillations according to type of experiment or observations being made. First, there are direct kinematic searches for neutrino mass and experiments looking for evidence of neutrinoless double beta decays. The present limits on the neutrino masses obtained from such studies, assuming the normal hierarchy of masses, are:

$$m_{\nu_2} < 190 \text{ keV} \quad \text{and} \quad m_{\nu_3} < 18.2 \text{ MeV}. \tag{71}$$

The best limits for the mass of the lightest neutrino, instead, have been obtained from the Mainz and the Troitsk experiments, where

$$m_{\nu_1} < 2.2 \text{ eV}. \tag{72}$$

The search for neutrinoless double beta decays is important because the observation of these decays would be a clear indication in favor of a Majorana neutrino, assuming CPT invariance holds. The Heidelberg-Moskow collaboration published a result in 2001 and later in 2004 showing at least a 2.2σ signal for neutrinoless double beta decay [17], but this result has been strongly contested and its validity is still an open question.

The second group of experiments are designed to investigate neutrino fluxes from natural sources: solar and atmospheric. These observations are historically important for being providing the earliest evidence for neutrino mixing and they remain highly relevant in astrophysical and cosmological research.

Lastly, we also look at experiments that employ neutrino beams produced at accelerators and nuclear reactors. They are usually divided in long- and short-baseline, according to the distance between the neutrino production point and the detector. Many short baseline accelerator experiments weren't able to detect any signal for neutrino oscillations. Nonetheless, they are important, because they provide us with constraints on the possible values of the neutrino mixing parameters. The most important limits have been obtained by NOMAD and CHORUS at CERN, which had experiments designed to detect τ production from a ν_μ beam, indicating ν_μ–ν_τ oscillations [24].

A. Neutrino-less double beta decay

Double beta decay is a higher-order nuclear process wherein a nuclei with charge Z changes into another with $Z+2$ without modifying its atomic mass A. In nature, there are 35 isotopes known to have the right ground state configuration for the decay reaction

$$(Z, A) \to (Z+2, A) + 2e^- + 2\overline{\nu_e}, \tag{73}$$

which can be seen as two simultaneous neutron beta decays. Another decay mode for double beta decay was proposed by Giulio Racah in 1937 and Wendell Furry in 1939:

$$(Z, A) \to (Z+2, A) + 2e^-, \tag{74}$$

a process which clearly violates lepton number conservation and is forbidden by the Standard Model. It can be viewed as a two-step process

$$\begin{aligned} (Z, A) &\to (Z+1, A) + e^- + \overline{\nu_e}, \\ (Z+1, A) + \nu_e &\to (Z+2, A) + e^- \end{aligned} \tag{75}$$

where a neutron first undergoes ordinary beta decay and then if $\nu_e = \overline{\nu_e}$ indicating we have Majorana neutrinos, then the second process can occur after that. Moreover, to allow helicity matching, it is necessary to have $m_\nu > 0$ so that the neutrinos don't possess fixed helicities.

For more detail, let's consider the matrix element for the decay,

$$\mathcal{M} \sim 2G_F^2 W^{\mu\nu} \left[\overline{u}\gamma_\mu P_L \left(\frac{-i\gamma_m u p^\mu + m}{p^2 + m^2} \right) P_L \gamma_\nu^T \overline{u}^T \right] \tag{76}$$

where $\overline{u}, \overline{u}^T$ are external electron spinors and $W^{\mu\nu}$ is the weak current matrix element. The factor $P_L\gamma_\nu^T\overline{u}^T$ comes from antiparticle considerations. Because

$$P_L \gamma^m u P_L = 0 \tag{77}$$

only the mass term in the middle factor contributes, giving rise helicity flips and $\Delta L \neq 0$.

For small neutrino mass, the decay rate is suppressed by a factor $|U_{ej}^2 m_j|^2$, which represents a factor of at least 10^{12} smaller in probability compared to its 2ν counterpart . Despite being partially compensated by other factors such as having a larger phase space to work with, this tiny relative probability of neutrinoless double beta decay makes it hard to observe experimentally.

The neutrinoless mode can be distinguished experimentally from the 2ν mode by examining the energy spectra of the daughter electrons. In neutrinoless double beta decay

$$E_1 + E_2 = Q, \tag{78}$$

where $Q \approx 2$ MeV is the typical decay energy for the mode, whereas in the 2ν case,

$$E_1 + E_2 < Q \tag{79}$$

since there is a distribution of electron energies depending on how much energy goes to the neutrinos. To observe this difference, we would need to examine the electron double beta decay spectrum near its end point at high energy resolution. The neutrinoless decay mode is also sensitive to the different CP-violating phases $e^{i\alpha}$ that appear in the charged current interaction for Majorana neutrinos.

Experimentally, the search for neutrinoless double beta decay relies on finding a peak in the region below 4.3 MeV. Common to all experiments is the low-background environment due to long expected half-lives. The types of experiments being made are

1. Ge semiconductor devices: Heidelberg-Moscow collaboration, IGEX;
2. Cd-Zn-Te detectors: COBRA in the Gran Sasso Underground Laboratory;
3. Cryogenic bolometers: CUORINCO at Gran Sasso;
4. time-projection chambers: NERMO-3 in the Frejus Underground Laboratory.

So far, there is no compelling evidence to suggest that neutrinoless double beta decay does indeed occur. Estimates on the decay lifetime lead to corresponding limits on the neutrino mass, and some of the key experimental results available are shown in table (II).

Isotope	Half-life (yrs)	$m_\nu^{\rm limit}$(eV)
${}^{48}_{20}$Ca→${}^{48}_{22}$Ti	9.5×10^{21}	8.3
${}^{76}_{32}$Ge→${}^{76}_{34}$Se	1.9×10^{25}	0.35
${}^{76}_{32}$Ge→${}^{76}_{34}$Se	0.7×10^{25}	0.6
${}^{82}_{34}$Se→${}^{82}_{36}$Kr	2.1×10^{23}	2.3
${}^{100}_{42}$Mo→${}^{100}_{44}$Ru	5.8×10^{23}	1.2
${}^{116}_{48}$Cd→${}^{116}_{50}$Sn	1.7×10^{23}	1.7
${}^{128}_{52}$Te→${}^{128}_{52}$Xe	7.7×10^{24}	1.1
${}^{130}_{52}$Te→${}^{130}_{54}$Xe	3.0×10^{24}	1.0
${}^{136}_{54}$Xe→${}^{136}_{56}$Ba	4.4×10^{23}	2.3
${}^{150}_{60}$Nd→${}^{150}_{62}$Sm	2.1×10^{21}	4.1

TABLE II. Decay lifetime estimates and the corresponding neutrino mass limits.

Chain	Reaction	$\Phi_\oplus$ (cm^{-2}s^{-1})
pp	pp	5.95×10^{10}
	pep	1.40×10^{8}
	hep	9.30×10^{3}
	^{7}Be	4.77×10^{9}
	^{8}B	5.05×10^{6}
CNO cycle	^{13}N decay	5.48×10^{8}
	^{15}O decay	4.80×10^{8}
	^{17}F decay	5.63×10^{6}

TABLE III. Solar process that produce neutrinos and the corresponding fluxes.

B. Solar neutrinos

Neutrinos play an essential role in stellar evolution because of reactions such as

$$4\text{p} \to \alpha + 2e^+ + 2\nu_e + 28\ \text{MeV} \tag{80}$$

which represents a series of successive processes that lie at the core of neutrino astronomy. Because neutrinos hardly react with any intervening matter, it is more advantageous to examine solar neutrinos as opposed to solar radiation, since the former carry valuable information about the core of stars, allowing us indirect access to whatever happens in there. The main processes responsible for neutrino production [8] are listed in table (III).

The earliest study of solar neutrinos was the Homestake experiment [19] headed by Raymond Davis, Jr. and John Bahcall in the late 1960s, which used the inverse decay on chlorine

$$^{37}\text{Cl} + \nu_e \to \ ^{37}\text{Ar} + e^-, \tag{81}$$

with threshold energy $E_{\rm thr} \approx 0.81$ MeV. Hence, it was sensitive to the pep, ^{7}Be, ^{8}B and hep components of the solar neutrino flux. The results were really surprising, because Homestake found a discrepancy in the solar neutrino flux of more than 60% that predicted by the standard solar model [20]. The updated value of the double ratio R for the chlorine experiment,

$$R = \frac{(\mu/e)_{\rm data}}{(\mu/e)_{\rm MC}}, \tag{82}$$

where MC stands for Monte Carlo predictions of the ratio (μ/e) of muon-electron events, is

$$R = 0.34 \pm 0.03. \tag{83}$$

The advantage of considering (μ/e) is that the flux uncertainties do not affect the result. If there are no neutrino oscillations, then $R = 1$ [21].

The Homestake result was later verified in similar experiments using gallium, which has a lower energy threshold ($E_{\rm thr} \approx 233$ keV), making them sensitive to the main pp component of the solar neutrino flux. The gallium results indicated that

$$\begin{aligned} R &= 0.60 \pm 0.05\ \text{(SAGE)}, \\ R &= 0.58 \pm 0.05\ \text{(GALLEX, GNO)}. \end{aligned} \tag{84}$$

These observations confirmed the existence of the solar neutrino puzzle.

Essential improvement in solar neutrino observations came with the water Cerenkov experiments, Kamiokande, and Super-Kamiokande that probed into the elastic scattering $\nu_e + e^- \to \nu_e + e^-$ and confirmed the electron neutrino deficit [18]. The real breakthrough, however, with the SNO deterium Cerenkov experiment in 2001, which studied three processes simultaneously: charged-current, neutral current and elastic scattering reactions. During its first phase, SNO observed the charged current and elastic scattering events, with an energy threshold for electron detection of 6.75 MeV. The flux measured from the charged current was

$$\Phi_{\nu_e,\rm C} = 1.75 \pm 0.07 \times 10^6\ \text{cm}^{-2}\text{s}^{-1} \tag{85}$$

which corresponded to a double ratio of

$$R = 0.35 \pm 0.03. \tag{86}$$

The observations at Super-Kamiokande didn't agree with the ν_e flux found at SNO, which found

$$\Phi_\nu = 3.69 \pm 1.13 \times 10^6\ \text{cm}^{-1}\text{s}^{-1}. \tag{87}$$

This result was the first evidence (at 3σ level) of the presence of different flavored neutrinos in the solar neutrino beam, providing the most robust evidence of into other active neutrinos. It is also remarkable that the sum of the ν_e and ν_μ fluxes give a total flux value that agrees well with Standard Model predictions, which actually strongly disfavors the hypothesis that these active neutrinos oscillate into sterile ones.

Experiment	Double ratio R
Kamiokande	$0.60 \pm 0.06 \pm 0.05$
IMB	$0.54 \pm 0.05 \pm 0.12$
Sudan2	$0.69 \pm 0.19 \pm 0.09$
Frejus	0.87 ± 0.21
NUSEX	0.99 ± 0.40

TABLE IV. Double ratios for atmospheric neutrino observations.

Super-Kamiokande	sub-GeV	$0.638 \pm 0.16 \pm 0.050$
	multi-GeV	$0.658 \pm 0.030 \pm 0.078$

TABLE V. Best available data for R.

C. Atmospheric neutrinos

Atmospheric neutrinos are useful both in the direct study of neutrino oscillations and as the background and calibration beam in the search for neutrinos from astrophysical sources [22]. Interactions of cosmic ray protons with nuclei in the upper atmosphere leads to various processes that produce electron and muon-type neutrinos that belong to the following chain of reactions:

$$\begin{aligned} p + X &\to \pi^{\pm} + Y \\ \pi^{\pm} &\to \mu^{\pm} + \nu_\mu \\ \mu^{\pm} &\to e^{\pm} + \nu_e + \nu_\mu \end{aligned} \tag{88}$$

where we don't distinguish between neutrinos and antineutrinos, as what is done in experiments. The $\pi^{\pm}$ can be replaced by $K^{\pm}$. A simple counting argument shows that $r = \nu_\mu/\nu_e = 2$. The values for the double ratio R in early experiments (82) can be found in table (IV). The error bars account for the statistical and systematic errors, respectively. The first two experiments in the list employ Cerenkov detectors while the last three used iron bar calorimeters. In the water Cerenkov detectors, the muons are distinguished from the electrons either (i) by the size of the Cerenkov light rings or (ii) by observing the products of muon decay.

The flux observations for atmospheric neutrinos were verified later on with experiments by the Super Kamiokande collaboration in the 1990s, which obtained the best known results for the double ratio, shown in table (V).

Notice that the first three experiments in table (IV) implies either muon depletion or an excess of electron events. To decide which, we can examine the up-down symmetry of detection events. Up (U) refers to events where the neutrino crossed the earth before hitting the detector while down (D) refers to events where the neutrino directly comes from the atmosphere to the detector. This leads to an observable quantity

$$\alpha_\ell = \frac{U - D}{U + D}. \tag{89}$$

Experiments indicate that for the electron, $\alpha_e = 0$ while $\alpha_\mu < 0$ for high momenta. This evidence suggests that $R < 1$ is a consequence of a reduced muonic neutrino flux, which could be due to oscillations whose probabilities are enhanced by the interaction with matter.

D. Reactor experiments

Nuclear reactors have long played a pivotal role in neutrino research, from their first detection to the most recent neutrino oscillation experiments. Reactors are high intensity, isotropic sources of ν_e, a product of β^- decay of neutron-rich fragments of uranium-plutonium fission. Almost all reactor experiments detect antineutrinos via inverse beta decay

$$\overline{\nu_e} + p \to e^+ n. \tag{90}$$

The observed energy spectrum for inverse beta decay has a peak around 3.6 MeV. Inverse beta decay occurs only for electron antineutrinos with sufficiently high energies ($\gtrsim 1.8$MeV) so only roughly a quarter of the fission events with $\overline{\nu_e}$ is detected from the reactor core.

When Reines and Cowan first detected the neutrino, they were looking for antineutrinos from a nuclear reactor [23]. In an initial experiment in 1953, they saw an antineutrino signal from inverse beta decay as the delayed coincidence between a positron and neutrino capture in cadmium, releasing ~ 8 MeV in gamma rays. Because the experiment had a high background(signal-to-noise: 1/20) the results were inconclusive: 0.41 ± 0.20 events/minute.

In 1956 the experiment at Savannah River yielded better observations. There, the inverse beta decay produced two signals: (i) a prompt signal from the positron annihilation, creating 2 gamma rays with energy 0.51 MeV and (ii) a delayed neutron capture with the ~ 8 MeV gammas. With a signal-to-background ratio of 3 to 1, they reported a cross-section of

$$\sigma = 6.3 \times 10^{-44}\text{cm}^2 \tag{91}$$

which agreed with the theory at the time. In 1959 parity violation was found and Reines and Cowan revised their analysis to include parity violation effects and arrived at

$$\sigma = 12^{+7}_{-4} \times 10^{-44}\text{cm}^2 \tag{92}$$

which matched the new theoretical predictions.

As was established before, neutrino oscillations are elegantly described by a picture where flavor eigenstates ν_ℓ are viewed as admixtures of mass eigenstates ν_j, which undergo mixing according to the unitary matrix (42). The PMNS matrix is typically expressed in terms of Euler angles $\theta_{12}, \theta_{13}, \theta_{23}$ and depending on model, may include CP-violating phases α_i as well.

The oscillation probability (40) shows the θ dependence, which in almost all neutrino studies typically involve the disappearance of electron or muon neutrinos.

For reactor studies, experiments are often designed to be most sensitive to the parameter θ_{13}, that is, lightest to heaviest mass transitions.

The best current limit on $\sin^2 2\theta_{13}$ comes from the CHOOZ collaboration [25]. The CHOOZ experiment uses a single detector with a 5 ton target of GF-loaded liquid scintillator at a distance 1.05 km from two 4.2 GW_{th} reactors. It observed a prompt signal from the positron and delayed signal from neutron capture that yielded $\kappa = 4, 5$ gammas with energies ~ 8 MeV:

$$n +^m \text{Gd} \rightarrow \ ^{m+1}\text{Gd}^* \rightarrow \text{Gd} + \kappa\gamma. \tag{93}$$

The Gd serves to shorten the neutron capture time to 300 μs, increasing the neutron signal while simultaneously reducing background effects.

The ratio of observed to predicted flux is

$$r = 1.01 \pm 0.028(\text{stat}) \pm 0.027(\text{sys}). \tag{94}$$

For $\Delta m^2 = 2.5 \times 10^{-5}$ eV^2, the CHOOZ result corresponds to

$$\sin^2 2\theta_{13} < 0.15 \qquad \text{at 90\% confidence level.} \tag{95}$$

E. Accelerator-based studies

The pioneering accelerator experiments on neutrino oscillations were designed to look for oscillations similar to quark mixing, that is for small mixing angles. CHORUS and NOMAD are two recent experiments that found no oscillations between ν_μ and ν_τ, setting the oscillation probability limit to below 1 percent. There have also been experiments (K2K, MINOS) that unambiguously measured the disappearance of ν_μ, which would correspond to $\Delta m^2 = 3 \times 10^{-3}$ eV^2.

The accelerator LNSD experiment has reported evidence of muon to electron neutrino transition at $\Delta m^2 = 0.1$ eV^2. If this result is verified, it would call into question the current number of neutrinos and strongly suggest the existence of other neutrino flavors. What is known at the moment is that the LNSD signature can be excluded as being due to a simple oscillation scenario, as shown by the MiniBooNE experiment.

One of the key issues in accelerator neutrino studies wishes to address is the existence of sterile neutrinos, Solar and atmospheric evidence already suggest that two of the three mixing angles are large. The third mixing angle hasn't been measured but the results from the CHOOZ experiment imply that it has to less than 8°. If the third mixing angle is non-zero, it follows that there will be CP-violation in the lepton sector.

In general, neutrino beams originate from charged particles that decay into neutrinos. To isolate the neutrinos, the charged particles are magnetically focused to travel in one direction and given enough space in which to decay while being in flight. Conventional beams start with pions that yield muon neutrinos or antineutrinos, with a small electron neutrino component from kaon and muon decay. The π themselves are obtained by first producing an intense beam of protons then putting a long, thin target material in the path of the protons—long so the interaction time for creating π is increased while thin so that secondary interactions of the π with the target are reduced.

Detector	M(kton)	ν_e?	ν_μ?	ν_τ?	E_ν^{ideal}
Liquid Ar	0.6	Y	Y	-	huge
H_2O Cereknov	50	Y	Y	-	< 2 GeV
Emulsion/Pb/Fe	0.27	Y	Y	Y	> 0.5 GeV
Scintillator	1.0	Y	Y	-	huge
Steel	5.4	-	Y	-	> 0.5 GeV

TABLE VI. Detector types used or planned in accelerator neutrino studies. M indicates largest mass to date. ν_ℓ? indicates event identification.

Focusing a neutrino beam requires a strong magnetic field with an integrated path length proportional to the radius at which the particles enter the focusing system. The standard design for a focusing system features a parabolic horn: an inner conductor shaped like a parabola holding the magnetic field between inner and outer conductors such that the length of the field at radius r is proportional to r^2. Typical currents running through such a horn are $\sim 10^5$ A.

Conventional neutrino beams are produced containing ν_μ and a small contamination of ν_e. To measure $\nu_\mu \rightarrow \nu_e$ transitions, we must distinguish between not only between electrons and muons but also between electrons and π^0, which constitute a large fraction of neutrino interactions at $\gtrsim 100$ MeV. Neutrinos that come from the decay of muons present an entirely different challenge because we should also keep track of the muon charge—a flip in the charge indicates $\nu_e \rightarrow \nu_\mu$—so it is crucial that the muon charge is not misidentified. There are various detectors used to observe neutrinos and we summarize the most important ones for current and future experiments in table (VI).

With beamlines and detectors in place, oscillation probabilities are measured based on the following relation:

$$N_{\text{far}} = \Phi_{\nu_\mu} \sigma_{\nu_\ell} \Pr[\nu_\ell|\nu_\mu] \epsilon_\ell M_{\text{far}} + B_{\text{far}} \tag{96}$$

whereN_{far} is the number of events at a far detector, Φ_{ν_μ} is the muon neutrino flux, σ_{ν_ℓ} is the ν_ℓ cross-section, M_{far} is the far detector mass, ϵ_ℓ is the detector efficiency for signal ℓ and B_{far} is the predicted background events.

To reduce systematic uncertainties, it is conventional to place an additional detector close to the neutrino source, at a distance before oscillations can occur. Then one predicts the number of events in the far detector in terms of the ratio between the near and far detector background events:

$$B_{\text{far}} = N_{\text{near}} \frac{S_{\text{far}}}{S_{\text{near}}} \tag{97}$$

Experiment	ν_μCC	NC	ν_e	ν_τCC	Signal	F
K2K (2006)	0	1.3	0.4	0	1	0.6
MINOS (2006)	5.6	39	8.7	4.7	29.1	3.1
OPERA (2003)	1	5.2	18	4.5	10	1.6
T2K (2001)	1.8	9.3	11.1	0	103	9
NOvA (2004)	0.5	7	11	0	148	11.5

TABLE VII. Signal and background rates for typical long baseline ν_e search. νCC, ν_e and NC refer to various possible background events. F is the figure of merit for sensitivity to oscillation events, defined as signal divided by the square root of sum of signal and background events.

Parameter	Best fit	2σ	3σ
Δm^2_{12} (10^{-5} eV^2)	7.6	7.3 – 8.1	7.1 – 8.3
$\lvert\Delta m^2_{13}\rvert$ (10^{-3} eV^2)	2.4	2.1 – 2.7	2.0 – 2.8
$\sin^2\theta_{12}$	0.32	0.28 – 0.37	0.26 – 0.40
$\sin^2\theta_{23}$	0.50	0.38 – 0.63	0.34 – 0.67
$\sin^2\theta_{13}$	0.007	≤ 0.033	≤ 0.050

TABLE VIII. Three-flavor neutrino oscillation parameters from global analysis of solar, atmospheric, reactor (KamLAND and CHOOZ) and accelerator (K2K and MINOS) experiments. Includes best fit estimate and 2σ and 3σ ranges.

where $S_j = \sum_\ell \Phi_{\nu_\ell}\sigma_{\nu_\ell}\epsilon_{\ell\ell'}M_j$ for $j = \text{near}, \text{far}$ and $\epsilon_{\ell\ell}$ describes the efficiency of identifying ν_ℓ as the signal $\nu_{\ell'}$.

The most important information gathered from accelerator-based experiments has been on $\nu_e \to \nu_\mu$ oscillations. The standard exposure for optimal analysis of electron neutrino appearance in a muon neutrino beam is shown in table (VII). The signal rates indicated assume $\sin^2\theta_{13} = 0.1$ and $\Delta m^2 \sim 2.75 \times 10^{-3}$ eV^2.

F. Summary of current experimental data

The developments in neutrino oscillation experiments in the last few years has given us a rough picture of the parameters governing three-flavor oscillations: There are two squared mass differences separated by a factor $\approx$ 30, there are two large mixing angles (θ_{23}, which could be as much as 45°, and θ_{12}, which is large but almost certainly smaller than 45°, according to data known at high significance level), and one mixing angle which must be small (θ_{13}). Present data is consistent with 2 possible ordering of masses for neutrinos, typically parameterized by the sign of Δm^2_{13}:

1. In the normal hierarchy ($\Delta m^2_{13} > 0$) the mass state which contains predominantly ν_e has the smallest mass;
2. In the inverted hierarchy ($\Delta m^2_{13} < 0$), ν_e is part of a nearly degenerate doublet of mass states which is separated from the lightest neutrino mass by $|\Delta m_{13}|$.

A global analysis of presently available neutrino oscillation data is depicted in fig. (1) and summarized in table (VIII).

V. MODELS INCORPORATING NEUTRINO MASS

The Standard Model is based on the gauge group $SU(2)_\mathrm{L} \times U(1)_\mathrm{Y}$. But this only fixes the gauge bosons in the theory—fermions and Higgs content are chosen somewhat arbitrarily. The choice made in the Standard Model corresponds to massless neutrinos but evidence for neutrino oscillations tells us neutrinos must possess mass. If we stay with the same gauge group, it is still possible to conjecture extra fermions or Higgs bosons that will predict massive neutrinos. This provides us with the simplest extensions of the Standard Model and is usually achieved via the see-saw mechanism.

Of course, one may choose to expand the gauge group into something where symmetry breaking leads to the Standard Model gauge. This is achieved in models attempting to unify all forces at short distance, the so-called grand unified theories, the most popular of which involve $SU(5)$, $SO(10)$ and E_6. Due to space limitations, we will briefly cover only how neutrino masses are generated using the first two gauge groups.

For completeness, we mention that there is also a class of Standard Model extensions that lead to non-vanishing neutrino masses by introducing a symmetry between bosons and fermions known as supersymmetry, which is desirable because it provides a direct solution to the hierarchy problem in the Standard Model.

A. The see-saw mechanism

One strange property of the Standard Model is that it includes left and right chiral projections of all fermions except neutrinos. Thus, it seems that a natural way to extend it is to add N right-handed neutral fields $N_{\ell R}$ with zero hypercharge and which are assumed to be $SU(2)_\mathrm{L}$ singlets. These fields have no interaction with gauge bosons but will have non-trivial effects due to their other properties.

The simplest model considers $N_{\ell R}$ to be the right-handed components of a Dirac neutrino:

$$\nu_\ell = \begin{pmatrix} \nu_{\ell L} \\ N_{\ell R} \end{pmatrix}. \tag{98}$$

The $N_{\ell R}$ fields give rise to extra mass terms in the Lagrangian

$$-\mathcal{L}_M = \sum_{\ell,\ell'} M_{\ell\ell'}\overline{\nu_{\ell L}}N_{\ell' R} + \text{(h.c.)} \tag{99}$$

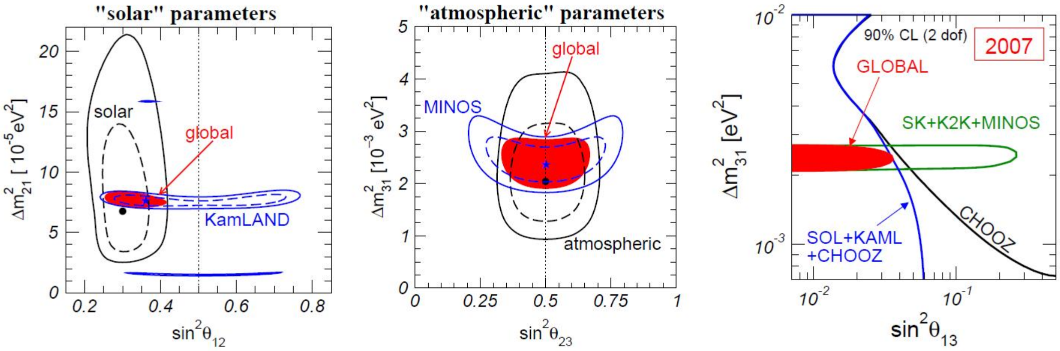

FIG. 1. Best 'solar' (left) and 'atmospheric' (middle) neutrino oscillation parameters determined from a combined analysis of various neutrino data sources as of 2007. Also, a global analysis of the neutrino data established some constraints on θ_{13} (right). Image courtesy of [26].

where $M_{\ell\ell'} = \left(v/\sqrt{2}\right) f_{\ell\ell'}$ is an $N \times N$ mass matrix. In general, $M_{\ell\ell'}$ isn't diagonal in the flavor basis so that the fields $\nu_{\ell L}, N_{\ell' R}$ do not correspond to physical fermion fields.

The physical fields are obtained from the eigenvectors of M. If we diagonalize M such that

$$U^\dagger M V = m, \qquad m = \mathrm{diag}(m_1, ..., m_N) \tag{100}$$

then we can write

$$\nu_{\ell,L} = \sum_\alpha U_{\ell\alpha} v_{\alpha L}, \quad \nu_{\ell,L} = \sum_\alpha U_{\ell\alpha} v_{\alpha L}. \tag{101}$$

This allows us to rewrite eq. (99) as

$$\mathcal{L}_M = \sum_\alpha \overline{\nu_{\alpha L}} m_\alpha \nu_{\alpha R} + (\text{h.c.}), \tag{102}$$

which shows that ν_α are the mass eigenstates with mass m_α.

Neutrino mixing follows from eq. (4) for our Dirac neutrino here when we write it in terms of the mass eigenstates:

$$-\mathcal{L}_C = \frac{g}{\sqrt{2}} \sum_\ell \sum_\alpha \overline{\ell^+_L} \gamma^\mu U_{\ell\alpha} \nu_{\alpha L} W^-_\mu + (\text{h.c.}) \tag{103}$$

which shows an expression analogous to quark mixing.

The problem with the Dirac neutrino model is that it makes no restrictions on the size of $f_{\ell\ell'}$, which consequently means it doesn't guarantee that neutrino masses will be small. This can be fixed by considering instead Majorana neutrinos. If we also introduce the fields $\hat{N}_{\ell L}$ conjugate to $N_{\ell R}$, then we get extra bare mass terms

$$-\mathcal{L}_B = \frac{1}{2} \sum_{\ell,\ell'} B_{\ell\ell'} \overline{\hat{N}_{\ell L}} N_{\ell' R} + (\text{h.c.}), \tag{104}$$

which with the identity

$$\overline{\nu_{\ell L}} N_{\ell' R} = \overline{\hat{N}_{\ell' L}} \hat{\nu}_{\ell R} \tag{105}$$

leads to a mass matrix of the form

$$\mathcal{M} = \begin{pmatrix} 0 & M \\ M & B \end{pmatrix}. \tag{106}$$

The matrix $\mathcal{M}$ has $2N$ Majorana neutrinos in general since M and B are $N \times N$ matrices.

To see further implications, suppose we look at the simplest case, $N = 1$. Then assuming $M, B \in \mathbb{R}, B > 0$, we can choose some rotation matrix

$$O = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \tan 2\theta = \frac{2M}{B} \tag{107}$$

such that

$$O\mathcal{M}O^T = \begin{pmatrix} -m_1 & 0 \\ 0 & m_2 \end{pmatrix} \tag{108}$$

and

$$m_{1,2} = \frac{1}{2}\left(\sqrt{B^2 + 4M^2} \mp B\right). \tag{109}$$

This isn't quite the diagonalization we want because of the minus sign so if we introduce $K = \mathrm{diag}(i, 1)$ then we can write

$$\mathcal{M} = O^T m K^2 O. \tag{110}$$

Introducing column vectors

$$\begin{pmatrix} n_{1L} \\ n_{2L} \end{pmatrix} \equiv O \begin{pmatrix} \nu_L \\ \hat{N}_L \end{pmatrix}, \qquad \begin{pmatrix} n_{1R} \\ n_{2R} \end{pmatrix} \equiv K^2 O \begin{pmatrix} \hat{\nu}_R \\ N_R \end{pmatrix} \tag{111}$$

we get a mass term involving $\mathcal{M}$ in the Lagrangian that can be reduced into

$$-\mathcal{L}_M = m_1 \overline{n_{1L}} n_{1R} + m_2 \overline{n_{2L}} n_{2R} + (\text{h.c.}) \tag{112}$$

where $n_1 = -\hat{n}_1,\ n_2 = \hat{n}_2$, showing n_1 and n_2 are Majorana neutrinos. In general, we get $2N$ Majorana neutrinos.

This model now explains why active neutrino masses should be small. Recall that M comes from Higgs coupling and so it is natural to assume it is the same order of magnitude as the other fermions in the same generation, i.e. the charged leptons in the case of neutrinos. Suppose $B \gg M$, then

$$m_1 \approx \frac{M^2}{B}, \qquad m_2 \approx B, \tag{113}$$

from which it follows that $m_1 \ll M$. Thus, if the active neutrino masses are small, the sterile neutrino masses have to be big in compensation. This is exactly the see-saw mechanism at work.

However, there are cosmological arguments which limit the mass of any stable neutrinos to $\lesssim 1$ eV. If we believe ν_τ to be stable then we can take $M \sim m_\tau$ and find

$$B \gtrsim 5 \times 10^9 \text{GeV}. \tag{114}$$

Comparing this to the weak scale of $\sim 10^2$ GeV, we see a huge gap in energy scales. This is known as the *hierarchy problem* and it appears in any conventional grand unified theory.

B. Expanding the Higgs sector

If no new fermions are included in the Standard Model, then we only have two degrees of freedom that correspond to uncharged fermions $\nu_{\ell L}, \hat{\nu}_{\ell R}$, which if they have mass must necessarily be Majorana particles. This also implies $B - L$ violation. $B - L$ can be recovered is new Higgs bosons are introduced to compensate for the $B - L$ due to neutrinos.

Given the lepton fields

$$\psi_L = \begin{pmatrix} \nu_{\ell L} \\ \ell_R \end{pmatrix} \tag{115}$$

with antiparticles given by

$$\hat{\psi}_R = \gamma_0 C \epsilon \psi_L^* \quad (2,1), \qquad \hat{\ell}_L = \gamma_0 C \ell_R^* \quad (1,2) \tag{116}$$

where $\epsilon = i\sigma_2$ is a 2×2 antisymmetric matrix, we can form fermion bilinears that have non-vanishing $B - L$ quantum numbers:

$$\overline{\psi_L} \hat{\psi}_R \quad (1,2) \oplus (3,2), \qquad \overline{\hat{\ell}_L} \ell_R \quad (1,-4). \tag{117}$$

The Higgs multiplets that can directly couple to these bilinears to form gauge invariant Yukawa couplings are

1. triplet field $\vec{\Delta} \quad (3,-2)$,
2. single-charge singlet $(1,-2)$, and
3. double-charge singlet $(1,4)$.

With $B - L = -2$ the electric charge of the triplet $\vec{\Delta}$ is

$$\vec{\Delta} = \begin{pmatrix} \Delta^0 \\ \Delta^- \\ \Delta^{-2} \end{pmatrix} \tag{118}$$

and additional Yukawa coupling outside those already in the Standard Model:

$$-\mathcal{L}_Y = \sum_{\ell,\ell'} f_{\ell\ell'} \overline{\psi_{\ell L}} \frac{1}{\sqrt{2}} \vec{\sigma} \cdot \vec{\Delta} \hat{\psi}_{\ell' R} + (\text{h.c.}) \tag{119}$$

where $\vec{\sigma}$ denotes the vector of Pauli matrices.

The Higgs potential now involves both $\vec{\Delta}$ and the usual doublet ϕ. Let us assume that the parameters in this potential are such that its minimum corresponds to

$$\langle \phi_0 \rangle = \frac{v_2}{\sqrt{2}}, \qquad \langle \Delta_0 \rangle = \frac{v_3}{\sqrt{2}}, \tag{120}$$

which after symmetry breaking leads to the mass term

$$-\mathcal{L}_M = \sum_{\ell,\ell'} \overline{\nu_{\ell L}} M_{\ell\ell'} \hat{\nu}_{\ell' R} + (\text{h.c.}) \tag{121}$$

where $M_{\ell\ell'} = v_3 f_{\ell\ell'}/\sqrt{2}$.

Using the conjugate field we can rewrite eq. (119) as

$$-\mathcal{L}_Y = \sum_{\ell,\ell'} f_{\ell\ell'} \hat{\psi}_{\ell R}^T C^{-1} \epsilon \frac{1}{\sqrt{2}} \vec{\sigma} \cdot \vec{\Delta} \hat{\psi}_{\ell' R} + (\text{h.c.}). \tag{122}$$

In this form, it is clear that the bilinear involves $\hat{\psi}_R$ twice and therefore it must obey Fermi statistics. Note that $f_{\ell\ell'} = f_{\ell'\ell}$ makes the mass matrix symmetric.

The mass eigenvalues and eigenstates are obtained from diagonalizing M. The couplings $f_{\ell\ell'}$, however, are unspecified so the definite pattern of neutrino mixing can't be uniquely determined. In this case, neutrinos are light because $v_3 \ll v_2$. This can be seen from the observation that since Δ_0 couple to W and Z

$$\rho \equiv \frac{M_W^2}{M_Z^2 \cos\theta_W} = \frac{1 + 2(v_3/v_2)^2}{1 + 4(v_3/v_2)^2} \tag{123}$$

where using experimental bounds on ρ:

$$\frac{v_3}{v_2} < 0.07. \tag{124}$$

Going back to the question of $B - L$ symmetry, the Yukawa coupling eq. (119) as such does not show $B - L$ violation. However, once this quantum number is carried by $\vec{\Delta}$, it is broken when Δ_0 acquires a non-vanishing vacuum expectation value. In fact, it breaks $B - L$ by 2 units, exactly what is needed for generating the Majorana mass terms.

Next we consider the second option: a model with a single-charged singlet h^- first proposed by Anthony Zee in 1980 [29], which involves the Yukawa coupling

$$-\mathcal{L}_Y = \sum_{\ell,\ell'} f_{\ell\ell'} \overline{\ell_L} \hat{\psi}_{\ell' R} h^- + (\text{h.c.}). \tag{125}$$

Because h has an electric charge, its vacuum expectation value must vanish, leading to a prediction of massless neutrinos. The situation changes, however, if there are more than one Higgs doublets in the theory. For instance, if there is a second doublet ϕ' then the trilinear coupling

$$\mu\phi^T\epsilon\phi' h + (\text{h.c.}) \tag{126}$$

yields the $B-L$ violation we want.

The mass terms are particularly simple if we assume that only one of the Higgs doublets couples to leptons. In this case,

$$M_{\ell\ell'} = \kappa f_{\ell\ell'}\left(m_\ell^2 - m_\ell'^2\right) \tag{127}$$

where the coupling constant is $f_{\ell\ell'} = -f_{\ell'\ell}$ so M is symmetric.

Interestingly, this model comes with a pattern of neutrino mixing, unlike the previous models discussed above. If we define

$$\tan\alpha = \frac{f_{\mu\tau}}{f_{e\tau}}\left(1-\frac{m_\mu^2}{m_\tau^2}\right), \qquad \tau = \frac{f_{e\mu}}{f_{e\tau}}\frac{m_\mu^2}{m_\tau^2}\cos\alpha, \tag{128}$$

then the mass matrix M in the flavor basis is

$$M = m_0\begin{pmatrix} 0 & \tau & \cos\alpha \\ \tau & 0 & \sin\alpha \\ \cos\alpha & \sin\alpha & 0 \end{pmatrix} \tag{129}$$

where $m_0 = \kappa m_\tau^2 f_{\tau e}/\cos\alpha$. Diagonalization of M here gives the physical masses; unfortunately, the predictions of the model don't agree with the most updated oscillation data.

The last option we consider is a model with a double-charge singlet, which can have Yukawa couplings of the form

$$-\mathcal{L}_Y = \sum_{\ell,\ell'} F_{\ell\ell'}\overline{\hat{\ell}_L}\ell'_R k^{+2} + (\text{h.c.}). \tag{130}$$

The field k^{+2} has a $B-L$ quantum number of 2 but it turns out that no amount of additional Higgs doublets will lead to the $B-L$ violation needed to generate neutrino masses.

What is required is the addition of a single-charge scalar of the same nature as h above [28]. With h^- and k^{+2}, we have a trilinear coupling

$$\mu h^- h^- k^{+2} + (\text{h.c.}) \tag{131}$$

which breaks the $B-L$ symmetry. Neutrino masses are generated in this scenario from 2-loop diagrams, which give rise to a mass matrix

$$M_{\ell\ell'} = 8\mu\sum_{\ell_1,\ell_2} m_{\ell_1}m_{\ell_2}f_{\ell\ell_1}F_{\ell_1\ell_2}f_{\ell'\ell_2}I_{\ell_1\ell_2} \tag{132}$$

where

$$\begin{aligned} I_{\ell_1\ell_2} &= \int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\left(\frac{1}{p^2-m_{\ell_1}^2}\right)\left(\frac{1}{q^2-m_{\ell_2}^2}\right) \\ &\times\left(\frac{1}{p^2-m_h^2}\right)\left(\frac{1}{p^2-m_h^2}\right)\left(\frac{1}{(p-q)^2-m_k^2}\right) \end{aligned} \tag{133}$$

so that the eigenvalues of M are naturally small because of the suppression factors in such two-loop integration terms.

An interesting prediction from this model follows from the asymmetry of $f_{\ell\ell'}$: $\det(M) = 0 \implies m_3 = 0$ if there are three generations. The third mass isn't necessarily zero because there are higher-order perturbation terms to consider but at this level it already suggests that $m_3 \ll m_1, m_2$ for three neutrino mass eigenstates.

C. Grand unified theories

Evidence for new physics such as dark matter and dark energy show some of the limitations of the Standard Model. Various attempts at extending it include grand unification theories (GUTs) that imagine scenarios wherein matter and energy are unified at some fundamental energy scale and where this overarching symmetry is spontaneously broken at the low energy scales where the Standard Model holds. Here we briefly examine how GUTs incorporate neutrino masses in the two most popular grand unification groups, $SU(5)$ and $SO(10)$.

In the Georgi-Glashow $SU(5)$ model, the Standard Model gauge groups are combined into a single gauge group $SU(5)$. Fermions are assigned to *10* and $\overline{5}$ representations that can be denoted by

$$F = \begin{pmatrix} \hat{d}_1 \\ \hat{d}_2 \\ \hat{d}_3 \\ e \\ \nu \end{pmatrix}_L, \quad T = \begin{pmatrix} 0 & \hat{u}_3 & -\hat{u}_2 & u_1 & d_1 \\ -\hat{u}_3 & 0 & \hat{u}_1 & u_2 & d_2 \\ \hat{u}_2 & -\hat{u}_1 & 0 & u_3 & d_3 \\ -u_1 & -u_2 & -u_3 & 0 & e^+ \\ -d_1 & -d_2 & -d_3 & -e^+ & 0 \end{pmatrix}_L \tag{134}$$

where $\hat{x}$ denotes the conjugate of x, e.g. $\hat{u}_L$ is the CPT-conjugate state to the right-handed helicity state of the up quark.

The breakdown of the gauge symmetry to $SU(3)_C \times U(1)_Q$ is achieved by two Higgs multiplets $H \equiv \overline{5}, \Phi \equiv$ *24* with the vacuum expectation value of Φ is chosen so that

$$\langle\Phi\rangle = \text{diag}\left(V, V, V, -\frac{3}{2}V, -\frac{3}{2}V\right) \tag{135}$$

where V is the unification scale obtained from unifying the three gauge couplings at low energies.

If we write the gauge bosons of $SU(5)$ as

$$\begin{pmatrix} \frac{1}{\sqrt{2}}\sum_a\lambda^a G_a + \sqrt{\frac{2}{15}}B_{24} & \mathcal{Z} \\ \mathcal{Z}^{-1} & \frac{1}{\sqrt{2}}\vec{\sigma}\cdot\vec{W} - \sqrt{\frac{3}{10}}B_{24} \end{pmatrix} \tag{136}$$

where

$$\mathcal{Z} = \begin{pmatrix} X_1^{\frac{4}{3}} & Y_1^{\frac{1}{3}} \\ X_2^{\frac{4}{3}} & Y_2^{\frac{1}{3}} \\ X_3^{\frac{4}{3}} & Y_3^{\frac{1}{3}} \end{pmatrix}, \quad \mathcal{Z}^{-1} = \begin{pmatrix} X_1^{-\frac{4}{3}} & X_2^{-\frac{4}{3}} & X_3^{-\frac{4}{3}} \\ Y_1^{-\frac{1}{3}} & Y_2^{-\frac{1}{3}} & Y_3^{-\frac{1}{3}} \end{pmatrix} \tag{137}$$

we get that

$$M_X^2 = M_Y^2 = \frac{25}{8} g^2 V^2. \tag{138}$$

The final stage to get to $U(1)_{\mathrm{Q}}$ occurs via H:

$$\langle H \rangle = \left(0, 0, 0, 0, \frac{v}{\sqrt{2}}\right)^T \tag{139}$$

which leads to

$$M_W = \frac{1}{2} g v, \qquad M_Z = \frac{g v}{2 \cos \theta_W} \tag{140}$$

where θ_W is the Weinberg angle. With a single gauge coupling constant, the unification scale is predicted to correspond to

$$\tan \theta_W = \sqrt{\frac{3}{5}}. \tag{141}$$

However, it turns out that at the scale M_Z, experimental data show that the three gauge couplings don't extrapolate to the same point at high energy scales. $SU(5)$ is not a viable GUT model unless there are more Higgs bosons introduced, which would ruin the simplicity and predictive power of the theory. Nonetheless, $SU(5)$ has proven to be an excellent training ground for understanding generic features of GUT models.

The most general gauge-invariant Yukawa coupling in the $SU(5)$ model is

$$\mathcal{L}_Y = h_1 T_{ij}^T C^{-1} F^i H^j + h_2 \epsilon^{ijklm} T_{ij}^T C^{-1} T_{kl} H_m + \text{(h.c.)} \tag{142}$$

Once the Higgs boson H acquires a vacuum expectation value, we get

$$M^d = M^\ell = h_1 \frac{v}{\sqrt{2}}, \qquad M^u = h_2 \frac{v}{\sqrt{2}} \tag{143}$$

where $M^{u,d}$ are the quark masses matrices and M^ℓ is for charged leptons.

At this stage, neutrinos remain massless. To modify the model to generate neutrino mass, a *15*-dimensional Higgs boson $S_{ij} = S_{ji}$ is included. The multiplet S allows the coupling

$$\mathcal{L}'_Y = f F i^T C^{-1} F_j S^{ij} + \text{(h.c.)} \tag{144}$$

Under the Standard Model gauge group, S contains the triplet scalar $\vec{\Delta}$. Assigning a non-zero vacuum expectation to S,

$$\langle S_{55} \rangle = \frac{u}{\sqrt{2}} \simeq \frac{\mu_S v^2}{V^2} \tag{145}$$

would naturally explain small neutrino masses when $\mu_S \sim O(V)$.

The next symmetry used in GUTs is $SO(10)$, first discovered by Howard Georgi a few hours before finding the $SU(5)$ model in 1973, and independently developed by Harald Fritzsch and Peter Minkowski in 1974. An elegant aspect of $SO(10)$ is that it contains the left-right symmetric gauge group $SU(2)_{\mathrm{L}} \times SU(2)_{\mathrm{R}} \times SU(4)_{\mathrm{C}}$ and thus, it automatically contains a right-handed neutrino, Furthermore, because quarks and leptons belong to the same irreducible representation, neutrinos acquire masses naturally through the same mechanism as the quarks and other leptons.

The spinor representation of $SO(10)$ is the *16*-dimensional $SO(10)$ which has a maximal subgroup $SU(4)_{\mathrm{C}} \times SU(2)_{\mathrm{L}} \times SU(2)_{\mathrm{R}} \times D$ where D is a discrete symmetry that essentially plays the role of charge conjugation on fermion fields. Known as D-parity, this symmetry maps $f_L \to \hat{f}_L$, interchanging the $(2,1,4)$ and $(1,2,\bar{4})$ sub-multiplets of the $SO(10)$ spinor. It plays an important role in understanding neutrino mass in left-right symmetric models.

Symmetry breaking is usually achieved with a combination of Higgs fields: choose $\mathit{45}_H$ or $\mathit{54}_H$ and then add either $\mathit{16}_H$ and $\overline{\mathit{16}}_H$ or $\mathit{126}_H$ and $\overline{\mathit{126}}_H$. Details are too involved to present here so the interested reader is referred to [31].

Now since $\mathit{16} \otimes \mathit{16} = \mathit{10} \oplus \mathit{120} \oplus \mathit{126}$, the mass-generating Higgs bosons must belong to $\mathit{10}, \mathit{120}$ or $\overline{\mathit{126}}$ dimensional representations of $SO(10)$. Fermion mass in $SO(10)$ originate from the Yukawa coupling

$$\begin{aligned} \mathcal{L}_Y = \quad & \sum_{a,b} f_{\mathit{10}}^{ab} \psi_a^T B C^{-1} \Gamma^i \psi_b \mathit{10}_i \\ & + f_{\mathit{120}}^{ab} \psi_a^T B C^{-1} \Gamma^i \Gamma^j \Gamma^k \psi_b \mathit{120}_{ijk} \\ & + f_{\mathit{126}}^{ab} \psi_a^T B C^{-1} \Gamma^i \Gamma^j \Gamma^k \Gamma^l \Gamma^m \psi_b \mathit{126}_{ijklm} + \text{(h.c.)} \end{aligned} \tag{146}$$

where Γ^i and B are the $SO(10)$ analogs of Dirac matrices and the spinor conjugation matrix, respectively. Note that f_{10}^{ab} and f_{126}^{ab} are symmetric while f_{120}^{ab} is symmetric. At the unification scale $M_{ab}^u = M_{ab}^d = M_{ab}^\ell = M_{ab}^0$.

Adding a *126* Higgs boson with a non-vanishing vacuum expectation value to its $SU(5)$ singlet component gives a Majorana mass to the right-handed neutrino:

$$\mathcal{L}_M^{N_R} = f_{\mathit{126}} \left(\frac{M_{BL}}{g}\right) N_R^T C^{-1} N_R \tag{147}$$

where it follows that the corresponding term for the left-handed neutrino is

$$\mathcal{L}_M^{\nu_L} = \lambda f_{\mathit{126}} \frac{k^2}{M_{BL}/g} \nu_L^T C^{-1} \nu_L \tag{148}$$

where k is the $SU(2)_{\mathrm{L}}$ breaking scale.

This leads to light neutrino mass given by

$$m_\nu \simeq \lambda f_{\mathit{126}} \frac{k^2}{M_{BL}/g} - \frac{1}{9} f_{\mathit{10}} f_{\mathit{126}}^{-1} f_{\mathit{10}} \frac{k^2}{M_{BL}/g} \tag{149}$$

which is just a different type of see-saw formula, where $m_\nu \to 0$ as M_{BL} is made very large.

Assuming f_{126} are roughly the same order of magnitude, it follows that

$$m_{\nu_\ell} \sim O(1 \text{ eV }), \qquad \ell = e, \mu, \tau. \tag{150}$$

On the other hand, if the direct Majorana mass vanishes ($\lambda = 0$) the model predicts, assuming $m_{\nu_\tau} < 18$ MeV, that

$$m_{\nu_\mu} \leq 1.5 \text{ keV}. \tag{151}$$

VI. CONCLUDING REMARKS

Despite the successes of the Standard Model, the phenomenon of neutrino oscillations presents the most convincing evidence for the need to modify it, because massless neutrinos can not oscillate. Put differently, observing neutrino oscillations imply neutrino masses cannot be equal—specifically, this means they cannot all be zero. Current experimental evidence suggests that if any neutrinos have non-zero mass, probably all of them do.

Experimental results show that the probability of a neutrino changing type is related to the distance between its point of production to its point of detection, with greater depletion for larger distances travelled. The oscillation probability is also a function of the neutrino masses and energy, and of the mixing angles, two of which are suspected to be relatively large and one much smaller than the other two.

Various phenomenological approaches have been proposed in extending the Standard Model into a more encompassing framework that incorporates neutrino masses, where the models discussed here involve more neutrinos, more Higgs bosons, or larger gauge symmetry groups. The hope is that future experiments in reactors and accelerators will achieve sub-percent precision for testing the neutrino mass hierarchy and establish which theory best describes the origin of neutrino mass.

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