Quantum Logic and Its Role in Interpreting Quantum Theory

Gelo Noel M. Tabia *

Perimeter Institute for Theoretical Physics,
31 Caroline Street North, Waterloo, Ontario, Canada N2L 2Y5

Institute for Quantum Computing and University of Waterloo,
200 University Avenue West, Waterloo, Ontario, Canada N2L 3G1

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Summary

Foundational studies in quantum theory have consisted in large part of attempts to avoid paradoxical situations in describing properties of a physical system. Quantum logic addresses some of the perplexing problems that plague interpretations of quantum theory by invoking a non-classical propositional calculus for the set of propositions of a quantum system.

Historically, quantum logic derives from von Neumann's observation that the set of projection operators on a Hilbert space constitute a 'logic' of experimental propositions. More than two decades passed from the pioneering work of Birkhoff and von Neumann before interest in quantum logic was re-ignited by Mackey's probabilistic analysis of quantum theory. Piron's axiomatization provided a significant extension of Mackey's formulation, leading to further developments focused primarily on establishing concrete operational foundations to quantum logic, most notably, the empirical framework introduced by Foulis and Randall in the 1980s. Recent advances in the past two decades feature the employment of powerful techniques from pure mathematics, in particular, various algebraic notions taken from category theory and computational logic.

Aside from establishing the formal structures characterizing the syntax of quantum logic, there have also been concerted efforts to attribute realist or empiricist readings to the semantic content embodied in the projection lattice of Hilbert space. Some prominent quantum-logical interpretations, at least those critically evaluated in this essay, include Reichenbach's trivalent logic, Putnam's realist quantum logic, and Finkelstein's operational quantum logic.

*Email: gtabia@perimeterinstitute.ca

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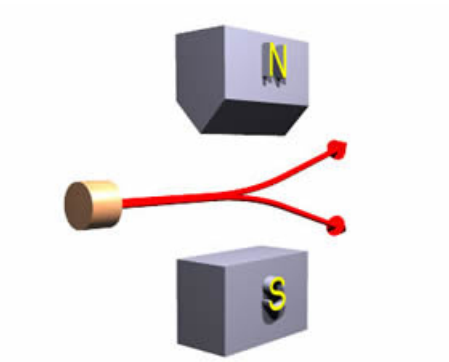

Figure 1: *Stern-Gerlach apparatus showing the splitting of a silver beam by the inhomogeneous magnetic field. Whether a particular silver atom is deflected up or down is determined by the spin of its unpaired electron. It is conventional to refer to the up and down portions as +1 and -1 outcomes, respectively. [Image courtesy of D. Harrison: UPSCALE, Department of Physics, University of Toronto.]*

1 The issue with quantum mechanics

It should not difficult to see why quantum mechanics is conceptually puzzling. The theory incorporates a computational method for assigning probabilities to events or propositions or values of physical magnitudes—the dynamical variables or 'observables' of the theory, formulated in terms of the geometry of Hilbert space. However, this particular algorithm has, over the years, resisted numerous attempts to associate the probabilities generated by the statistical quantum states with measures over the so-called 'property states'—that is, major problems arise when definite values are attributed to all relevant physical observables or to 'lists' of properties of the system, or when truth values are assigned to all possible propositions about the system [1].

To gain more insight into what the problem is, consider the mathematical framework of classical physics. In rather general terms, the algebra of physical observables of a classical system is a commutative algebra of real-valued functions defined on the position-momentum phase space of the system. It contains a sub-algebra of idempotent magnitudes (which correspond to projectors for classical dynamical variables) which is a Boolean algebra, isomorphic to the Boolean algebra of Borel subsets of phase space.[1] In this case, the property state of the classical system is represented by a point in phase space (an atom in the corresponding Boolean algebra) or by collections of sets to which the point belongs to (an equivalence class of propositions about the system) and the evolution of the system is usually described using Hamiltonian mechanics. It is also possible to define a statistical state for such a system, represented by a probability measure (in the usual Kolmogorov sense) over regions of phase space and where the time evolution of the state is captured by Liouville dynamics.

In direct comparison, a fundamental issue of interpretation is encountered in quantum mechanics: the theory apparently provides us with a set of states which are statistical states, expressed in terms of vectors or statistical operators in Hilbert space, without specifying any property states for the system. Moreover, there are excellent grounds (theorems by Bell, Kochen and Specker [2], for instance) for supposing that there are no such property states for quantum systems. The natural question to ask then is, what do these statistical states mean if there are no property states?

The following example [3] provides a concrete illustration of this peculiar problem: Think of a beam of silver atoms emitted by a source used in a Stern-Gerlach apparatus, like the one depicted in fig. (1). After the atoms pass through the inhomogeneous magnetic field (along the z-direction), the beam is split vertically and only two outcomes for σ_Z are found, corresponding to +1 or -1. If the apparatus had been horizontal (the counterfactual situation associated with σ_X), still only two outcomes are

[1] Roughly speaking, a Borel set is any set that can be formed from elementary sets and then taking countable unions and intersections of these sets. For example, the set of natural numbers $\mathbb{N}$ is a Borel set, since one can take each natural number as a set and the union of them is denumerable.

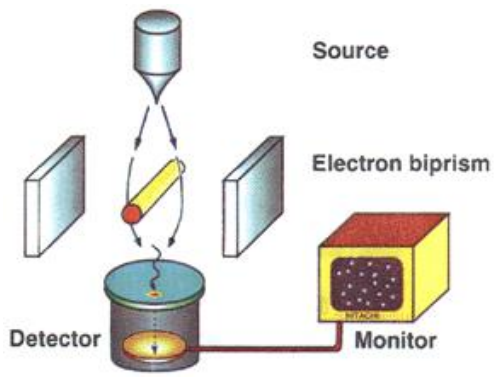

Figure 2: *Experimental arrangement of a two-slit experiment with electrons. Two paths are made available to a weak electron beam via the electron biprism, a interferometric device composed of parallel plates with a fine central filament, the filament having a positive electrostatic potential relative to the plates. [Image courtesy of A. Tonomura, et al.: American Journal of Physics* ***57*** *(1989) 117-120.]*

anticipated. For atoms that yielded outcome +1, it is true that

$$\sigma_Z = +1, \text{ and } \sigma_X = +1 \text{ or } \sigma_X = -1.$$

It is, however, never the case that

$$\sigma_Z = +1, \text{ and } \sigma_X = +1.$$

It is also false to say that

$$\sigma_Z = +1, \text{ and } \sigma_X = -1.$$

The last two statements are 'absurd' because they attribute simultaneous values to complementary observables, something which Bohr asserted to be meaningless to even talk about. However, the first claim is acceptable because only one of the non-commuting variables is given a fixed value.

Nonetheless, it is perfectly legitimate to say something about σ_X and σ_Z at the same time when referring to statistical ensembles of identically prepared systems. This is because, in practice, the selection and extraction of a sample from the ensemble remains a useful approximation to an idealized preparation process.

A similar situation happens when considering an electron two-slit experiment [4], shown in fig. (2). In analogy to the version of the double slit using photons, the electron paths are called 'slits'. Let X be any region on the detecting screen. Define the probabilities

$$\begin{aligned} p_1 &= P(X|\text{ only slit 1 is open }), \\ p_2 &= P(X|\text{ only slit 2 is open }), \text{ and} \\ p_{12} &= P(X|\text{ both slits are open }). \end{aligned} \tag{1}$$

Suppose there is another variable Y that takes values 1 or 2 depending on whether an electron passes through slit 1 or 2. Then it must be that

$$\begin{aligned} p_1 &= P(X|Y=1), \\ p_2 &= P(X|Y=2), \text{ and} \\ p_{12} &= P(X|Y=1 \text{ or } Y=2), \end{aligned} \tag{2}$$

which according to classical rules of conditional probabilities imply that p_{12} is some convex combination of p_1 and p_2, i.e., $p_{12} \in (p_1, p_2)$. The electron interference pattern seen in these experiments demonstrably violates this result. Note that although this second example talks about measurement statistics, the incorrect conclusion follows from ascribing definite, determinate values to incompatible properties of individual electrons. The usual thing to say is that it is not meaningful to talk about

which slit an electron went through if no attempt was made to observe it.

Quantum theory seems to leave us in an unsatisfying position of having to completely disregard statements about any unmeasured physical quantity whenever an observable incompatible to it has been measured—an unsavory, paradoxical consequence of Heisenberg's principle of indeterminacy. So this is the case for studying quantum logic: there are certain statements which are certified false by Boolean logic and meaningless according to the widely accepted version of quantum mechanics. Quantum logic stems from a desire to make such statements to be both meaningful and true [3].

One of the primary aims of quantum logic as a foundational program is to understand the meaning behind the formal structure of quantum theory by regarding it as a type of non-classical propositional calculus. [5] The motivation behind it can be described succinctly as follows: perhaps what is at fault in the present understanding of quantum mechanics is the insistence of a classical logical structure for describing properties or dynamical quantities of a system.

A possible solution to this prevailing dilemma would then involve the use of extended or generalized logics, which will take explicit account of situations involving incompatible observables. A skeptic of such a 'quantum logic program' might think this is nothing more than a needless obfuscation of well-known ideas but it should be noted that "difficulties like those with virtual processes or divergences in quantum field theory might be rooted in applying a formalism appropriate to measurable quantities and extending them to unmeasurable ones [6]."

2 Logic, philosophy, and interpretation

To fully appreciate what proponents of quantum logic are trying to achieve, it is first necessary to give a general overview of some important notions pertaining to logic and interpretations of physical theories.

2.1 Logic

Logic is a mathematical model for deductive thought. It is a model in much the same way as probability theory is a model for situations involving chance and uncertainty. A logical system is defined by a formal structure for constructing sentences, called syntax, and for attributing meaning to these sentences, termed semantics [7]. More intuitively, logic deals with propositions and relations and operations on those propositions, where the field of propositions that apply depends on the context being studied. Of course, this context must always be clearly specified from the start, in order to determine what valid, meaningful statements are admitted by the logical system.

As a mathematical model, logic is more abstract than any correspondence to real-life objects. Consider the following logically correct deduction:

"All men are mortal.
Socrates is a man.
Therefore, Socrates is mortal."

The validity of the third sentence (the conclusion) from the first two (the assumptions) does not depend on any special idiosyncrasies of Socrates. The deductive reasoning is justified by the form of the sentences rather than any actual fact about men and mortality. In fact, the meaning of the word "mortal" is unimportant here; however, it does matter what "all" means. The power of logic lies in the ability of making such inferences. Because the validity of logical deductions is determined by the form rather than the content, it is possible to analyze the syntax of sentences independent of the semantic meaning.

Formally, logical inference describes the relation between two propositions, an antecedent a and a consequence b, wherein $a \to b$ means that "if a is true then b follows". An even stronger statement would be the logical equivalence of propositions $a = b$, which is defined to be the pair of inferences $a \to b$ and $b \to a$. Inferences are governed by the axiomatic rules of the logical system, commonly referred to as the propositional calculus of the logic. Roughly speaking, the propositional calculus dictates how compound sentences and implications can be constructed from the starting axioms of the logic.

For propositional logic, the syntax sets the rules for building compound propositions out of elementary ones using operations called logical connectives:

Symbol	Logical operation	Also referred to as
$\neg$	NOT	negation
$\wedge$	AND	conjunction or 'meet'
$\vee$	OR	disjunction or 'join'
$\to$	IF-THEN	subjunction or implication
$\leftrightarrow$	IFF	equality

In classical Boolean logic, a simple way to give meaning to propositions and sentences is through the set-theoretic approach. Here the field of propositions is considered as a set S and every proposition is associated with a subset R of S. Thus, one gets a one-to-one correspondence between logical and set-theoretic operations: union and intersection of sets for the join and meet, respectively, and complements for negation of propositions. There is also a natural representation for the universal or 'evidently true' proposition 1 by the set S itself and the empty or 'patently absurd' proposition 0 by the null set $\emptyset$. More details about Boolean logic will be discussed in the section on the logic of classical mechanics.

2.2 Realism and empiricism

Scientists work under the impression that their field of study is more than just the mere accumulation of facts and observational data. A good scientist believes that a crucial aspect of his profession involves explaining why nature behaves the way that it does. As van Fraasen notes, "science does not merely represent phenomena but also interprets them" [8]. But what does it exactly mean to say that one "interprets" science?

For scientists, scientific interpretation is fundamentally concerned with understanding the conceptual framework or mathematical formalism that underlie natural processes. In order to clarify what is meant by 'understanding', it is first necessary to talk about what the aim of science is.

The scientific enterprise is a productive activity, one whose merits are measured in terms of the successes of its theories. The success of a scientific theory depends on its correspondence with natural phenomena in such a way that one is able to judge if the degree of correspondence is acceptable or believable. More generally, a theory is an object for the sort of attitudes that are widely accepted by the scientific community, expressed in assertions of knowledge and reasonable opinions. These knowledge and opinions are meant to tell us about what the world is like in a way that is not only logical but also predictive.

The reason for believing in a correspondence between a physical theory and the world around us falls under two broad philosophical views on the purpose of science. The first one is called realism. In contemporary philosophy, realism is the belief in the existence of certain objects independent of our thoughts, beliefs, or conceptions about them. A realist will typically profess to three fundamental beliefs [9]:

(a) Reality consists of everything that does exist. (This is meant to distinguish ontological or physical existence from Platonic realism.)

(b) Reality is independent of any act of observation.

(c) Some of the features of reality are accessible to our knowledge.

The last point is particularly important if we wish to avoid merely discussing pure metaphysics. In the realist perspective, a physical theory is then regarded as a theory of observable magnitudes, that is, as a theory which describes nature as nature reveals itself when it is probed by means of measurement [10]. It is assumed that nature exists independent of the process of observation but physical theory only requires some pertinent features of the world to be accessible under suitable conditions for observing them. Furthermore, a realist believes that the behavior of the unknown (that is, the unobserved or perhaps unobservable elements of nature) can be logically inferred from its (indirect) interaction with the known.

An objective approach to a physical theory must have in it, as an essential ingredient, the appropriate conditions for measurement, that is, the possible ways of observing a system must be defined in terms of primitive concepts in the theory, such as logic, causality, and space-time. These notions, which Bohm called 'explicit structures' of the part of nature being studied, together with a material basis for measurement processes that define physical quantities, determine the physical reality. In addition, a realistic theory is required to be complete in the sense of Einstein–that every element of physical reality correspond to an element of the theory. A final postulate concerns the internal consistency of a physical theory wherein the measuring process that underlies the foundations of the theory (since the theory is derived or deduced from such measurements) must be considered as physical processes that are contained within and subject to the complete theory itself. It might be worth mentioning here that any physical theory presupposes the validity of mathematical disciplines like logic, geometry, analysis, and algebra. These disciplines correlate well with experience. However, it must be maintained that logic, algebra, analysis and geometry do not have a priori validity for physics—hence, Wigner's [12] remarks about the "unreasonable effectiveness of mathematics" in describing natural phenomena.

As much as most scientists seek to describe reality by deducing its features from what is observed, there is, however, an alternative philosophy called empiricism. Empiricism in its most general form encompasses the belief that information about the world is acquired primarily through observation and experience. In its broadest sense, empiricism includes various schools of thought about what constitutes empirical reasoning, from instrumentalism and operationalism to logical positivism and constructive empiricism. The last one in particular emphasizes those aspects of knowledge that pertain to scientific evidence and offers a strong and modern contrary stance to realism and so let us say a few words about it.

Constructive empiricism states that scientific theories have literal meaning, that they aim to be empirically adequate, and that their acceptance involves, as belief, only that they are empirically adequate [8]. A physical theory is empirically adequate if and only if everything that it says about observable entities is true. It also means that nothing needs to be said about phenomena that isn't observed, and the same for unobservable quantities. A theory is regarded to have literal meaning or semantically literal if the concepts of the theory are expressed in such a way that it claims are necessarily true or false (as opposed to say an instrumentalist interpretation that a theory only needs to be able to explain and make accurate predictions). Constructive empiricism shares with realism the literal interpretation of a theory but differs from it in not ascribing any truth to unobservables.

2.3 Interpreting scientific theories

Most, if not all, interpretations of scientific theories can be classified as realist or empiricist. Despite major differences in content, the interpretation of scientific phenomena relies heavily on logic. The meaning attributed to propositions in that logic will depend on whether the interpretation gives a realist or empiricist reading, but in either case, inferences and deductions about how natural phenomena work will have to respect a formal logical structure, one which is intuitively simple in classical systems (since it is just a Boolean algebra) but more complicated in quantum systems (often termed the 'lattice of closed linear subspaces of Hilbert space', which will be described in detail in later sections).

It must now be said that even after one chooses to be a realist or empiricist, there are still a good number of ways to interpret the theory that will tally with its mathematical content. There must be some criteria one can use that will make one interpretation more attractive than another. As a start, scientists commonly adhere to what is called "Ockham's razor", which usually just means that between any two theories that explain the same phenomenon correctly, the simpler model is more preferable. It is a somewhat pragmatic criterion motivated by a straightforward logical reasoning: the more sophisticated objects a theory incorporates, the less likely each and everyone of them is actually realized, and so the less probable for that particular theory to be true. Of course, there are those people who argue that maybe the world really works in a complicated fashion and it is actually wrong to insist that it be simple; but throughout the years science has managed to come up with relatively simple yet effective models for describing chemical, biological, and physical processes of all kinds, and it seems more prudent to remain on that course.

According to Reichenbach [11], the world has a definite causal order although it is inherently unpredictable (due to quantum indeterminism). Therefore, he says a scientific theory should be judged according to its ability to provide a causal model for every correlation. Although not all correlations result from cause-effect links[2], the theory should still be able to suggest that most correlations are derived from a common cause (that is, it should not allow for too many pure coincidences). The point is that the more correlations a theory explains, the better the theory is. It seems to be a very reasonable criterion for an acceptable physical model of nature. Any scientific interpretation is expected to be able to explain strong correlations between physical quantities in terms of a direct causal connection.

Physicists in particular like to think of interpretation as being synonymous with physical understanding, which, roughly speaking, has to do with the 'visualizability' of concepts in the theory. A visualizable concept is one that admits a mental picture, often in terms of idealized objects with imaginable, concrete properties. Several examples worth citing are:

(a) In Newtonian mechanics, objects are thought of a point-like, rigid particles that experience forces, setting them into motion with trajectories that trace out a curve for any given time interval.

(b) In electromagnetic theory, a magnetic fields can be visualized as a set of field lines that are directed from the north to the south pole and whose densities in a particular region determine the flux strength there.

(c) In quantum information, the state of a qubit can be represented in terms of a point in the Bloch sphere or ball, where points on the surface correspond to pure states and points inside correspond to mixtures.

(d) In general relativity, space-time is imagined to be analogous to a rubber sheet that changes it shape in the presence of matter and energy, and whose shape or curvature determines the motion of the objects that move in it.

[2] As a comical example, consider the possible implications of a statistically significant inverse correlation between the number of pirates worldwide and the average global temperatures: fewer pirates, higher global temperatures $\Rightarrow$ pirates are cool. It would be true only if the correlation is causal in nature.

From the above examples, it is possible to characterize some features of visualizability. One way is to check if the system can be described in terms of idealized particles or waves. Something more general would need analogies of abstract ideas with more familiar objects such as rubber sheets or elastic springs. There also those concepts which find simple geometric counterparts—the sort of intuition advocated and popularized by Albert Einstein. In fact, modern advanced theoretical concepts are considered 'intuitive' if they can be depicted geometrically. It is essential that the fundamental notions of any interpretation involve visualizable concepts because there is no true understanding when a theory is mired in a web of rigorous yet incomprehensible formalities.

3 The logic of classical physics

To understand the differences in the conceptual structure between classical and quantum mechanics, it is instructive to first examine the notions of state of a system and observable quantities as they are studied in classical physics. Not only are classical realm more familiar and more intuitive, it will also be helpful to see how the quantum mechanical notions of states and observables depart from our classical expectations [13].

3.1 Classical states and observables

In the Hamilton-Jacobi formulation of classical analytical dynamics, a classical system consists of a set of particles whose individual motion has a rather simple geometric description. The physical properties of the system fall under two broad categories: some are like mass, which do not change with time, while others are like position, which vary with time. Thus, a complete characterization of the system and its behavior requires specifying the set Π_F of its constant or fixed-value properties and the set Π_V of its time-varying properties, at any particular time. There are also a set of laws Λ that govern the interactions of particles within the system among each other or with their environment.

Given Hamilton's canonical equations of motion, a particular time t, and the constant properties Π_F of the system, the values of all properties in Π_V of the system are in principle fully determined if the position $\vec{q}(t)$ and the momentum $\vec{p}(t)$ of each of the particles are known initially. It is for this reason that the classical state can be represented by a point $(\vec{q}(t), \vec{p}(t))$ that lies in the phase space Ω of the system. For example, if a single particle moving in one dimension has position q and momentum p, then the state of the particle is given by $\omega(q, p) \in \Omega$.

All relevant dynamical quantities are then just real-valued functions of these phase points. In mathematical terms, for every physical observable A there exists a function $f_A : \Omega \to \mathbb{R}$ such that for any state ω,

$$f_A(\omega) = f_A(q, p) \tag{3}$$

gives the value of A for that state. Thus, in the case of a one-dimensional particle of mass m,

$$T = f_T(q, p) = \frac{p^2}{2m} \tag{4}$$

represents its kinetic energy. In this case and that of most familiar dynamical quantities, there is a continuum of possible values. However, it is always possible to construct 'artificial' observables that takes only two values. Experimentally, such observables are in fact quite useful despite the apparent lack of specificity. For example, one might ask, "is the kinetic energy of the particle larger than 1 J?". This question corresponds to an observable which yields 1 if the answer is 'yes' and 0 if the answer is 'no'.

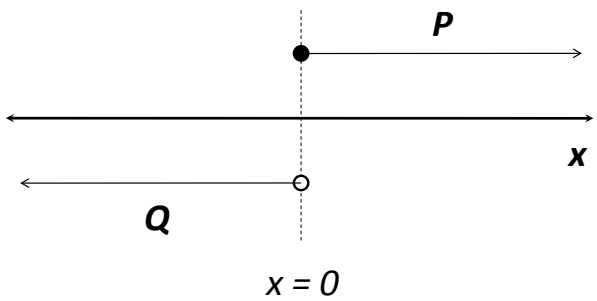

Figure 3: *Phase space of a very simple classical system, that of a coin which can be heads or tails. Each potential property is associated with a subset of phase space, in this case the two halves of the real number line.*

In general, the structure of any possible experimental two-valued question is something like, "does observable A have a value within the set Δ?" In which case, the state of the physical system of interest assigns to the question the value

$$\omega(A, \Delta) = 1 \qquad \text{iff} \qquad f_A(\omega) \in \Delta, \tag{5}$$

where $f_A : \Omega \to \mathbb{R}$ is the dynamical variable expressed as a phase-space function.

In classical mechanics, there are then two equivalent ways to think of the state of a system: (i)the former involving a sequence of coordinates of position-momentum phase space that essentially determines the system's properties at each instant, and (ii) the latter which regards the state as a binary function of the set of experimental questions that describe the disposition of the system to yield certain measurement outcomes. Later on it will be shown that the situation is very different for quantum mechanical systems. In particular, it seems that only the 'dispositional states' of the latter type are available; incompatible observables appear to suggest that it is impossible to assign property states to a system.

3.2 Boolean algebras

Common to all quantum logical approaches is the aim of providing an algebraic account of quantum theory. A great number of authors have sought to recapture the Hilbert space structure of quantum mechanics by looking at the algebraic constraints to which the property lattice or event structure of any quantum logical model must conform to. With that in mind, it is incumbent to provide a brief account of the corresponding algebraic structure in classical systems. Our discussion here will be similar to that of Hughes [13].

To illustrate the algebraic structure that coincides to the set of properties of a system, it is sufficient to explore a very simple classical system—a coin that is either heads or tails. For this two potential properties of the system, one can associate the real line where

$$\begin{aligned} P &= \text{set of points } x \geq 0, \\ Q &= \text{set of points } x < 0. \end{aligned} \tag{6}$$

as indicated in fig. (3). Note here that the coin is classical because it properties can be associated with subsets of a phase space, although in this case it isn't position-momentum phase space.

It is also possible to represent the relations between subsets of this phase space by drawing a graph where each node refers to a subset. Part of the network of all possible subset relations is illustrated in fig. (4). In the diagram, the top node corresponds to the entire space (the real line) while the

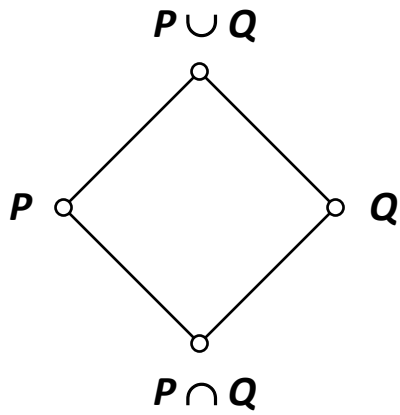

Figure 4: *Diagram for the subset relations of properties of the classical coin. The nodes refer to possible subsets and the lines denote proper subset inclusion of subsets in the lower nodes by sets in the higher nodes. For example,* $P \subset P \cup Q$*. In this simple example,* $P \cup Q = \mathbb{R}$ *and* $P \cap Q = \emptyset$*.*

bottom node is the null set. If any point in the graph can be reached from another by moving up along the lines, then the subset represented by the node in the upper position properly contains the subset represented by the node in the lower position. Thus, the lines in the graph actually represent the relation of subset inclusion, i.e. for any set A of a higher node and set B of a lower node, $B \subset A$ and B is strictly smaller than A. Because each node also represents a possible property of the system, the diagram also displays the relations among these properties.

For each property there is a corresponding sentence that expresses the fact that the system has the property associated with the node in the graph. For example, the sentence $\omega \in P$ says that "the coin is heads-up".[3] It is always possible to write compound sentences by combining elementary sentences such as "the coin is heads-up" or "the coin is tails-up" (or equivalently, " the coin is heads-up or tails-up"). This compound sentence is the same as stating that $\omega \in P$ or $\omega \in Q$. Such sentences are called propositions and if logical connectives are used then one denotes

$$\begin{aligned} p &= \omega \in P, \\ q &= \omega \in Q, \end{aligned} \tag{7}$$

then

$$p \vee q = \omega \in P \text{ or } \omega \in Q = \omega \in P \cup Q. \tag{8}$$

It is also possible to construct other sentences such as

$$\begin{aligned} \neg p &= \omega \in P^C \\ p \wedge q &= \omega \in P \cap Q, \end{aligned} \tag{9}$$

where P^C is the complement of P, which is defined whenever the set $U = P \cup P^C$ is given (in this case, U is the entire phase space but this is generally not the case). The network displaying all possible subsets of the space obtainable by taking unions and intersections of P, Q, P^C, Q^C is shown in fig. (5).

Each node also corresponds to an equivalence class of sentences that all say that the system has a particular property. The fact that there are many possible sentences is obvious when the system has properties whose values can be derived from more basic ones (i.e, one of them is a function of the

[3]For simplicity, the single quotation marks typically employed for delineating logical sentences is omitted for purely mathematical expressions.

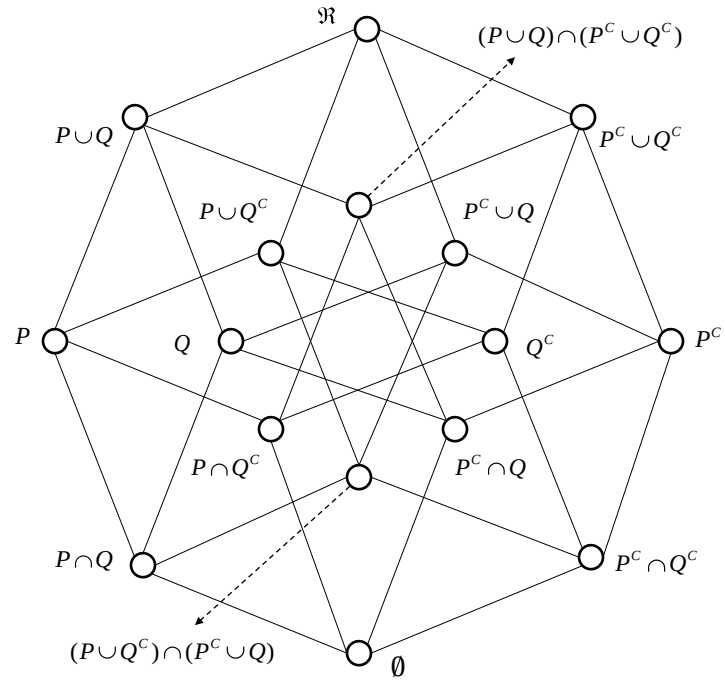

Figure 5: *The graph of all possible subset relations involving the sets P, Q, P^C, Q^C. The lines indicate subset inclusion of the lower nodes by the higher nodes to which they are connected. This 'lattice' structure represents a simple Boolean algebra.*

other). For example, the statement "the particle has momentum zero" has the same meaning as "the particle has zero kinetic energy" and can both be represented by the same subset or graph node. The equivalence class of such statements is what is properly called the 'propositions of the system'.

The discussion above tells us about the equivalence of the following sets of different objects:

(a) the set-theoretic relations among elements of a family of sets,

(b) the conceptual relations between members of a list of the relevant properties of a system; and

(c) the logical relations that hold between propositions belonging to various equivalence classes.

These sets are isomorphic to one another. Because they have the same algebraic structure, they are all represented by the same graph, which in the case of the coin example is shown in fig. (5). The common abstract structure represented by this diagram is an example of a Boolean algebra.

The algebra $\mathcal{B}$ is a Boolean algebra if $\mathcal{B} = \langle B, \vee, \wedge, \neg, 0, 1 \rangle$ where B is the set of elements that contain at least two elements designated as 0 and 1, $\vee$ and $\wedge$ are binary operations and $\neg$ is a unary operator on B, where the operations satisfy the following identities:

$$\begin{aligned} a \vee b &= b \vee a, & a \wedge b &= b \wedge a, \\ a \vee (b \vee c) &= (a \vee b) \vee c, & a \wedge (b \wedge c) &= (a \wedge b) \wedge c, \\ a \vee (a \wedge b) &= a, & a \wedge (a \vee b) &= a, \\ a \vee (b \wedge \neg b) &= a, & a \wedge (b \vee \neg b) &= a, \\ a \wedge (b \vee c) &= (a \wedge b) \vee (a \wedge c), & a \vee (b \wedge c) &= (a \vee b) \wedge (a \vee c). \end{aligned} \tag{10}$$

The list of postulates were chosen above to show the symmetry between $\vee$ and $\wedge$. The first two lines are just the commutativity and associative of the binary operations, the third line is called the 'absorption' postulate, the fourth line describe the properties of the $\neg$ operation and the last line shows

the distributivity between $\vee$ and $\wedge$. In this language, $\vee, \wedge$, and $\neg$ are known as 'join', 'meet' and 'negation', respectively.

It follows that for any $a, b \in B$

$$\begin{aligned} a \vee a &= a, & a \wedge a &= a, \\ a \vee \neg a &= b \vee \neg b, & a \wedge \neg a &= b \wedge \neg b, \\ \neg(\neg a) &= a. \end{aligned} \tag{11}$$

It says that there exists elements of B such that $a \wedge \neg a$ and $a \vee \neg a$ do not depend on the choice of a. These elements are defined as

$$0 \equiv a \wedge \neg a, \tag{12}$$

$$1 \equiv a \vee \neg a. \tag{13}$$

One can show that these elements also obey De Morgan's laws:

$$\begin{aligned} \neg(a \wedge b) &= \neg a \vee \neg b, \\ \neg(a \vee b) &= \neg a \wedge \neg b. \end{aligned} \tag{14}$$

The Boolean operations are all completely characterized by their action on $\{0, 1\}$, that is, they are defined completely by the elementary Boolean algebra $\mathcal{B}_2$. A consequence is that any Boolean algebra $\mathcal{B}$ can be homomorphically mapped onto $\mathcal{B}_2$. In other words, there exists functions which map $\mathcal{B}$ onto $\mathcal{B}_2$ that preserve the Boolean operations $\vee, \wedge$ and $\neg$. Formally,

Theorem 1. *For any Boolean algebra $\mathcal{B} = \langle B, \vee, \wedge, \neg, 0, 1 \rangle$, there exists functions $g : B \to \{0, 1\}$ such that for all $a, b \in B$*

$$\begin{aligned} g(a \vee b) &= g(a) \vee g(b), \\ g(a \wedge b) &= g(a) \wedge g(b), \\ g(\neg a) &= \neg g(a), \end{aligned} \tag{15}$$

where the operations $\vee, \wedge, \neg$ on the right-hand side are Boolean operators on $\mathcal{B}_2$.

The significance of mappings g for classical logic should be clear. By treating 0 and 1 as 'false' and 'true', the map g becomes a truth-functional that assigns truth-values to propositions of the Boolean algebra $\mathcal{B}$, which was previously demonstrated to be isomorphic to the list of properties of the system. It allows us to talk about the truth-value of propositions like $p \wedge q$ in terms of the truth-values of p and q.

It is also often said that classical systems exhibit a Boolean lattice. To rigorously define the lattice structure, a partial ordering relation must be specified. In set-theoretic language, this partial ordering relation is synonymous to subset inclusion. In terms of logical connectives, the ordering relation $\leq$ is properly specified by the following biconditional:

$$a \leq b \text{ iff } b = a \vee b \text{ (iff } a = b \wedge a \text{)}. \tag{16}$$

The relation $\leq$ of eq. (16) has the standard properties of partial ordering, i.e., for all $a, b, c \in B$,

$$\begin{aligned} \textbf{Reflexivity:} \quad & a \leq a, \\ \textbf{Transitivity:} \quad & a \leq b, b \leq c \Rightarrow a \leq c, \\ \textbf{Anti-symmetry:} \quad & a \leq b, b \leq a \Rightarrow a = b. \end{aligned} \tag{17}$$

This partial ordering is represented by the lines of the 'lattice' in fig. (5).

For completeness, let us make two more definitions, one about the idea of atomicity, the other about the important concept of ultrafilters.

Definition 1. *An element a is an atom of the Boolean lattice $\mathcal{B}$ if $a \neq 0$ and for all $a, b \in B$, $b \leq a$ implies $b = 0$ or $b = a$. Note that all finite Boolean algebras are atomic.*

Definition 2. *The ultrafilter $\mathcal{U}$ of an atomic Boolean algebra $\mathcal{B}$ is a set of elements of containing just one atom a and all points b such that $a \leq b$. More precisely, if $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$ then for all $a, b \in B$,*

$$\begin{aligned} a \vee b \in \mathcal{U} \quad & \textit{iff} \quad a \in \mathcal{U} \textit{ or } b \in \mathcal{U}, \\ a \wedge b \in \mathcal{U} \quad & \textit{iff} \quad a, b \in \mathcal{U}, \\ \neg a \in \mathcal{U} \quad & \textit{iff} \quad a \notin \mathcal{U}. \end{aligned} \tag{18}$$

There is a one-to-one correspondence between the set of ultrafilters on $\mathcal{B}$ with the set of homomorphisms $g_U : \mathcal{B} \to \mathcal{B}_2$. This correspondence enforces the truth-functional behavior of propositions in the Boolean lattice.

What has been shown so far is the isomorphism between a propositional lattice and some Boolean algebra. But note that Boolean algebras are purely structural: no special meaning needs to be attached to the operations. This means that it is possible to come up with other algebraic structures isomorphic to a Boolean lattice and completely abstract (no attached truth values or any semantic content for that matter). However, it is widely known that the set-theoretic realization has a special status: a representation theorem due to Stone [14] says that every Boolean algebra is isomorphic to a field of sets. This is significant in the context of classical physics because it means that it is more appropriate to think that because the logic of propositions about classical systems is Boolean, the propositions can be represented by subsets of a phase space. Most physicists are trained to think about this the other way around, i.e., that classical state phase space leads to a Boolean property lattice.

It is also worth noting that for classical systems, the graphs are all powers of two (so we can label them like G_{2^N}), all of which can be constructed from multiplication of the basic graph G_2, which has two vertices corresponding to 0 and 1, and one edge [3]. G_2 refers to a very basic proposition of the system and asks only whether such a property exists or not.

4 The logic of quantum physics

In 1936, Garrett Birkhoff and John von Neumann published a landmark paper [16] demonstrating how the logical structure of quantum theory is characterized by the lattice of closed linear subspaces of Hilbert space. This algebraic structure has some similar elements with Boolean algebras (for example, the sublattice of commuting observables is, in fact, Boolean) but where they differ clearly determines the point of departure of quantum logic from classical logic. To show their result, they have to postulate the dependence on Hilbert space at the onset. A major part of the quantum logical approach is to

reconstruct the same lattice structure from a primitive set of axioms that does not start with a Hilbert space.

Quantum logic has long been accused of being a notoriously difficult subject, mainly because a full appreciation for it requires knowledge of some profound and sophisticated logico-algebraic notions that are not normally encountered outside philosophy and advanced mathematics. In light of such difficulty, it behooves us to emphasize for a moment what benefits can be gained by going through all the seemingly unnecessary abstractions of quantum logical approaches.

Redhead [24] describes one unexceptionable way of thinking about what quantum logic is about. He said that the motivation of quantum logic is expressed by the following equation:

$$L + P' = L' + P, \tag{19}$$

where L refer to classical Boolean logic, the new paradoxical physics in quantum theory P', L is quantum logic and P is the sensible, intuitive physics of classical systems. In other words, the aim is to retain the more familiar classical picture in quantum mechanics by modifying the logic of quantum propositions from classical Boolean logic rather than changing the physics to accommodate the quantum results.

Let us look at a short historical overview of quantum logic. The discourse in this section follows closely the account in Cooeke, et al. [5]. It begins with a review of the important properties of the projection lattice of Hilbert space as formulated by Birkhoff and von Neumann.

4.1 Orthocomplemented projection lattices

John von Neumann's monumental treatise on quantum mechanics [17] established the most widely-accepted theoretical framework of the theory, in which each quantum system is associated with a Hilbert space $\mathcal{H}$, each unit vector $\psi \in \mathcal{H}$ specifies a possible state of the system, and each physical quantity associated with the system is represented by a self-adjoint linear operator $A \in \mathcal{L}(\mathcal{H})$. One can trace the origins of quantum logic with this particular formalization of quantum mechanics. Although it is conventional to think of physical properties in terms of the self-adjoint operators, it is actually more natural to associate quantum states with projection operators (for one, it eliminates the non-uniqueness of state vectors because of an undetermined overall phase). The argument is that it is the projection valued measure P_A, more than the operator A, that most directly carries the statistical interpretation of quantum mechanics. For instance, if $P \equiv P_A(B)$ is the spectral projection associated with an observable A and a Borel set B, one may construe this observable as "testing" whether or not A takes a value in B. In fact, von Neumann himself regarded P as representing a physical property of the system (or, rather of the state of the system).

To describe the projection lattice, it is helpful to first introduce several definitions relevant for subspaces:

Definition 3. *Given a vector space V, the set of vectors $S \subseteq V$ is called a subspace of V if and only if*

(a) $0 \in S$,

(b) if $|s_i\rangle \in S$ then $r = \sum_i c_i |s_i\rangle$ is some vector in S for scalars $c_i \in \mathbb{C}$.

Addition and scalar multiplication are inherited from the vector space V.

Definition 4. *The set $S^\perp$ of all vectors orthogonal to S is the orthocomplement of S. That is,*

$$S^\perp = \{|x\rangle : \langle x|y\rangle = 0, |y\rangle \in S\}. \tag{20}$$

Some important properties of orthocomplementation operator $\perp$:

$$\begin{aligned} (S^{\perp})^{\perp} &= S, \\ S \cap S^{\perp} &= 0, \\ S \oplus S^{\perp} &= 1. \end{aligned} \tag{21}$$

Definition 5. *For subspaces S, T, the set $S \cap T$ is called the meet of S and T. It is a subspace defined by*

$$S \cap T = \{|x\rangle : |x\rangle \in S, |x\rangle \in T\}. \tag{22}$$

Definition 6. *For subspaces S, T the smallest subspace containing both S and T is $S \oplus T$, called the join or linear span of S and T:*

$$S \oplus T = \{as + bt : s \in S, t \in T; a, b \in \mathbb{C}\}. \tag{23}$$

One useful way to think of the lattice of subspaces is to think of how it replaces the subsets of phase space in classical mechanics. Def. (4) is the analogue of complementation for sets, although a subtle difference results from how the orthocomplement $S^{\perp}$ involves only vectors orthogonal to S, whereas the complement P^C of P may involve any element outside of P. Def. (5) is essentially the same as taking intersections of sets but with subspaces. Def. (6) provides the biggest difference with sets, since the direct sum $S \oplus T$ is not at all the same as the union $S \cup T$. The direct sum or linear span includes all linear combinations of vectors in S and T, which is a much bigger set of vectors than just combining the elements of S and T.

The binary operations $\oplus, \cap$ and the unary operation $\perp$ correspond to logical connectives $\vee, \wedge, \neg$, respectively, for the set of quantum mechanical propositions. The propositional calculus associated with Hilbert subspaces equipped with a partial ordering relation between the closed subspaces—basically the same as subset inclusion if one thinks of subspaces as sets of vectors with some additional relations—defines the projection lattice, denoted by $\mathbb{P}(\mathcal{H})$.

Indeed, if P and Q are commuting projections, then their meet $P \wedge Q$ and join $P \vee Q$ in the lattice $\mathbb{P}(\mathcal{H})$ may be interpreted classically as representing the conjunction and disjunction of the properties encoded by P and Q. If P and Q do not commute, however, then they are not simultaneously measurable and the meaning $P \wedge Q$ and $P \vee Q$ is less clear. In particular, $\mathbb{P}(\mathcal{H})$ is orthocomplemented and so it enjoys analogues of the de Morgan laws for subspaces. Furthermore, the sub-lattice generated by any commuting family of projection operators is a Boolean algebra since commuting observables are effectively classical.

It is worth noting that while von Neumann talked of simultaneous measurability or testability of properties of a quantum system, he did not exactly distinguish between decidable and undecidable properties. Classically, any subset of the phase space counts as a categorical property of the system, and nothing in principle prevents us from taking a similar view in quantum mechanics. However, only closed linear subspaces of Hilbert spaces correspond to physical observables that are decidable by measurement. It is because of this lack of commensurability of all potential properties that the lattice $\mathbb{P}(\mathcal{H})$ of mutually orthogonal projections of a Hilbert space does not constitute a Boolean algebra.

The most distinguishing feature of $\mathbb{P}(\mathcal{H})$ is that it is a non-distributive lattice, that is, if A, B, C are distinct propositions about some physical properties then in general,

$$A \cap (B \oplus C) \neq (A \cap B) \oplus (A \cap C), \tag{24}$$

of which the spins along perpendicular directions in a Stern-Gerlach apparatus and the electron two-

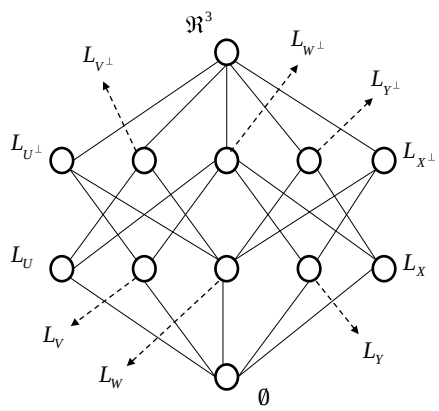

Figure 6: *The Greechie lattice G_{12} of subspaces of two triples $a = \{U, V, W\}$ and $b = \{X, Y, W\}$ of orthogonal vectors in $\mathcal{H}_3$. The two subset of subspaces corresponding to each triple ($L_{a,a^\perp}$ or $L_{b,b^\perp}$), plus the empty set and $\mathcal{H}_3$ separately form a distributive sub-lattice of eight elements. Since each subset is complemented, these sub-lattices are also Boolean algebras.*

slit experiment provide concrete examples. Hilary Putnam [18] characterized the difference between classical and quantum logic mainly by this failure of distributivity. However, this is a weak and even misleading characterization of the projection lattice—it is not merely non-distributive [1]. In terms of a Kochen-Specker type of analysis, quantum logic has a particular non-Boolean lattice: it is some sort of splicing together of Boolean algebras within a larger non-Boolean structure.

To have some idea of what type of lattice Hilbert subspaces generate, an example is given in fig. (6). This is the Greechie lattice G_{12} generated by subspaces of two distinct sets of three orthogonal vectors, $\{\vec{u}, \vec{v}, \vec{w}\}$ and $\{\vec{x}, \vec{y}, \vec{w}\}$, in a three-dimensional Hilbert space [13], where $L_{\vec{x}}$ denotes the ray spanned by $\vec{x}$ with orthocomplement $L_{\vec{x}^\perp}$, and so on. G_{12} is a partial Boolean lattice with two distributive sub-algebras generated by $L_{\vec{x}}, L_{\vec{y}}, L_{\vec{w}}$ and $L_{\vec{u}}, L_{\vec{v}}, L_{\vec{w}}$, which are 'pasted together' at the minimum and maximum elements.

4.2 Mackey's probability calculus

In a review article [19] written in 1957, George Mackey introduced the idea of treating probabilities associated with quantum events as a form of non-standard probability model, mainly by substituting in the projection lattice $\mathbb{P}(\mathcal{H})$ for the classical Boolean algebra of classical events. A re-derivation of the projection lattice is achieved from the premise that the logic of quantum experimental propositions is most accurately represented by $\mathbb{P}(\mathcal{H})$.

In Mackey's formulation, quantum states and observables can be expressed purely in the language of $\mathbb{P}(\mathcal{H})$ in the following manner:

Definition 7. *Any statistical quantum state Q determines a probability measure on $\mathbb{P}(\mathcal{H})$*

$$\omega_Q : \mathbb{P}(\mathcal{H}) \to [0, 1](P \mapsto \mathrm{tr}\,\{PQ\}), \tag{25}$$

where P is a projection operator in $\mathbb{P}(\mathcal{H})$. Gleason's theorem guarantees us that the probability measure on $\mathbb{P}(\mathcal{H})$ will have this unique form.

Definition 8. *Any physical observable A that takes values from the measurable space $\mathcal{A}$ may be represented by a projection-valued measure via the mapping*

$$M_A : \mathcal{A} \to \mathbb{P}(\mathcal{H}) \tag{26}$$

where for each set $B \in \mathcal{A}$, the projection $M_A(B)$ refers to the proposition "observable A yields an

outcome within the set B when measured".

To obtain probability measures on $\mathcal{A}$ probability measures on $\mathbb{P}(\mathcal{H})$ can be pulled back along the function M_A so that

$$\text{prob}(b \in B, m = M_A(b)|Q) = \omega_Q(M_A(B)) = \text{tr}\left\{M_A(B)W\right\}, \tag{27}$$

provided that the state of the system before measuring observable A is Q. A connection with von Neumann's representation of observables is then readily established: If $f : S \to \mathbb{R}$ is any bounded classical real-valued random variable defined on S then define the self-adjoint operator

$$A_f = \int_S f(s)dM(s) \tag{28}$$

so that for any probability measure μ on $\mathbb{P}(\mathcal{H})$, the expectation value of observable A is

$$\langle A_f \rangle = \int_s f(s)dM^*(\mu_Q(s)) = \text{tr}\left\{A_f Q\right\}, \tag{29}$$

where Q is the density operator corresponding to μ_Q. This is just the usual expression for quantum mechancial expectation values.

Although an algebraic structure fully isomorphic to the projection lattice is recovered from this model, it admittedly has one crucial ad hoc element: the Hilbert space $\mathcal{H}$ itself. Therefore, one still needs to explain why natural systems are modeled using projection operators on Hilbert spaces and not by some more general mathematical space. Mackey himself attempted to deduce a Hilbert space model by starting with an abstract structure $(\mathfrak{O}, \mathfrak{S}, p)$ where $\mathfrak{O}$ represent real-valued observables, $\mathfrak{S}$ represent physical states of the system, and p is a mapping

$$p : \mathfrak{O} \times \mathfrak{S} \to \delta : (A, s) \mapsto p_A(a|s), \tag{30}$$

where δ is the set of Borel probability measures. The intended interpretation is that $p_A(a|s)$ gives the probability distribution for measurement outcomes a of the observable A when the system is in the state $s \in \mathfrak{S}$. The pair (A, B) represents the experimental proposition that a measurement of observable A yields an outcome in the real Borel set B. In this formulation, two propositions are deemed equivalent if they produce the same probabilities in every state. The set L of experimental questions $P_{A,B} \equiv p_A(B|s) \forall s \in \mathfrak{S}$ defines the quantum logic. With point-wise partial ordering on $\mathfrak{S}$, the set L is an orthocomplemented, partially ordered set with unit 1 given by $P_{A,\mathbb{R}}$ for any observable A and whose orthocomplement $P^{\perp}_{A,B}$ is given by $P^{\perp}_{A,B} = 1 - P_{A,B} = P_{A,\mathbb{R}\setminus B}$.

Despite the considerable strengths of Mackey's framework, it does suffer from one major weakness: it takes probability as a primitive concept. It therefore inherits all the problems associated with the concept of probability, in particular, questions about interpreting probabilities: do they represent objective properties of a system, or degrees of knowledge or belief, or some predisposition of systems to yield particular measurement outcomes? There are some modern approaches developed around Mackey's formalism involving orthoalgebras that now have probability as a derived notion but these topics are beyond the level of this essay.

4.3 Piron's question-proposition system

It has been said that much of the structure of the projection lattice $\mathbb{P}(\mathcal{H})$ is reproduced in Mackey's probabilistic formulation. Nevertheless, the lattice of closed subspaces of Hilbert spaces has more

regularity built into it as opposed to the rather general orthocomplemented lattice one arrives at with Mackey's axiomatization. Piron's [20] efforts gets us closer to the projection lattice by employing a framework that considers a physical property to be determining factor in obtaining outcomes of experimental tests with certainty.

In constructing his axiomatic formalism, Piron was thinking about a realistic point of view, which takes the idea that a physical system has well-defined, pre-determined properties, whether or not the values of these properties are known by anyone. In such a scenario, it is straightforward to ask whether there are suitable experimental tests for measuring the value of any property. It is sufficient to consider a set of questions $\mathcal{Q}$ with binary outcomes. The state of the system P can be thought of as a preparation procedure that 'causes' the system to yield definitely affirmative outcomes for some particular tests. To go further, it will be useful to introduce the following definitions:

Definition 9. *A question $\alpha \in \mathcal{Q}$ is any experiment that generates outcomes corresponding to a 'yes' or 'no' response. If one can confirm that the outcome will always yield 'yes', then the question is said to be true for the system under investigation.*

Definition 10. *The set of all questions that are true for a given physical system defines its state. Note that because the state will generally evolve according to some dynamical law, the value of properties may change with time. Therefore, the truth value of a question is also a function of time.*

Definition 11. *A question α is deemed stronger than the question β if β is true every time α is true. This is a relation expressing a physical law which shall be expressed as*

$$\alpha \leq \beta. \tag{31}$$

This partial ordering relation $\leq$ defines an equivalence class of questions α, which are referred to as propositions. A proposition $[\alpha]$ is true whenever any question in the equivalence class is true. A true proposition can be identified with an actual property of the system while any other proposition refers to a potential or possible property of the system.

Let $\mathcal{L} \equiv \{[\alpha] : \alpha \in \mathcal{Q}\}$ be the set of all such propositions, considered as a partially ordered set under set inclusion $\subseteq$. Note that $[\alpha] \subseteq [\beta]$ if and only if every preparation making α certain necessarily makes β certain as well. There is a theorem that tells us about the structure of $\mathcal{L}$.

Theorem 2. *The set $\mathcal{L}$ of all propositions defined for a system and equipped with a partial ordering relation $\leq$ is a complete lattice.*

To show this, one only needs to show for any subset A of propositions a_i there exist propositions b and c in A such that

- given $a_i \leq b$, and if $a_i \leq d$ then $b \leq d$, the element $b = \vee a_i$ is called the supremum or least upper bound of A, while
- given $c \leq a_i$, and if $e \leq a_i$ then $e \leq c$, the element $c = \wedge a_j$ is called the infimum or greatest lower bound of A.

The symbols $\vee$ and $\wedge$ are used in anticipation of the fact that they will be directly related to the logical connectives $\vee$ and $\wedge$. There are two trivial propositions 0 and 1, which correspond to the minimal and maximal elements of propositions of the entire partially-ordered set. If one defines

$$\vee\, a_i = \wedge x, \qquad a_i \leq x \quad \forall i, \tag{32}$$

then it follows that $\wedge$ corresponds to $\wedge$. However, $\vee$ only corresponds to one direction of $\vee$ ($a \vee b \Rightarrow a \vee b$)

unless it can be shown that the lattice is distributive (which is not desired if one is attempting to reconstruct the projection lattice).

The state S of the system is completely defined by the true proposition $p = \wedge x, x \in S$ since

$$S = \{x : p \leq x, x \in \mathcal{L}\}. \tag{33}$$

Completeness of the lattice requires the hypothesis that p is an atom, which follows directly from this postulate:

Axiom 1. *If a is a proposition different from the trivial one 0 then a is true for some state and there exists an atom $p \leq a$.*

It is also necessary to define the orthocomplement $a^{\perp}$ for a:

Axiom 2. *For any $a \in \mathcal{L}$ let there be a dual proposition $a^{\perp}$ called the orthocomplement of a such that*

$$\begin{aligned} a' \vee a &= 1, \\ a' \wedge a &= 0, \end{aligned} \tag{34}$$

such that there exists questions $\alpha \in a$ and $\alpha' \in a'$ where α' is just the inverse question of α obtained by exchanging 'yes' and 'no'.

Note that there is also a theorem regarding the Boolean substructure of compatible propositions:

Theorem 3. *If for each state and each proposition $a = [\alpha]$ it is the case that either "a is true" or "$a^{\perp}$ is true" then $\mathcal{L}$ is Boolean and isomorphic to the lattice of all the possible subsets of some set.*

The projection lattice is a partial Boolean algebra for commuting observables so the theorem will apply for the sub-lattice of compatible propositions in $\mathcal{L}$ but not the entire lattice itself.

Finally, although the lattice is not distributive, it has a weaker sort of distributivity called orthomodularity:

Axiom 3. *The lattice $\mathcal{L}$ is orthomodular if for $a, b, a^{\perp}, b^{\perp} \in \mathcal{L}$, where $a^{\perp}, b^{\perp}$ are the orthocomplements of a, b, respectively,*

$$a \leq b \Rightarrow a^{\perp} = b^{\perp} \vee (a^{\perp} \wedge b^{\perp}). \tag{35}$$

Summarizing, it is possible to introduce just a few additional axioms in order to make Piron's question-proposition system $\mathcal{L}$ a complete, atomistic, orthocomplemented lattice satisfying the covering law:

(a) **Completeness:** By taking into account product questions and demanding closure under formation of arbitrary product questions lead to $\mathcal{L}$ being closed under arbitrary intersections.

(b) **Orthocomplement:** There exists an inverse question $\alpha^{\perp}$ to α defined by interchanging the roles of the dichotomic alternatives. What is required is $[\alpha] \cap [\alpha^{\perp}] = \emptyset, [\beta^{\perp}] \cap [\alpha] = 1$ for some $\beta \in [\alpha]$.

(c) **Orthomodularity:** This essentially introduces distributivity for compatible questions.

(d) **Atomicity and the Covering Law:** This is imposed in a somewhat ad hoc fashion but with substantial physical motivation.

Now it must be said that Piron's axioms lead to a lattice $\mathcal{L}$ that is isomorphic to the lattice of closed subspaces of a generalized Hilbert space. (Because of orthomodularity, the quantum Hilbert space is not uniquely singled out.) A more serious limitation of Piron's axioms can be found when considering the composition of two or more separate systems. For example, if one has a bipartite system where each subsystem conforms to the axioms individually, the combined system as a whole conforms to the axioms if and only if one of the subsystems is classical in nature [21].

4.4 Foulis-Randall operational framework

David Foulis and Charles Randall [22] synthesized ideas from their dissertations on abstract lattice theory and concrete operational statistics, respectively, in order to develop what is termed empirical quantum logic. Their formalism is based on the primitive notion of an operation or test–that is, a well-defined mutually exclusive alternative possible outcomes. Although this idea pertains to possible experiments that can be performed, similar to Piron's formalism, it is different in that instead of treating the questions or propositions as basic elements, here the outcomes are the building blocks of the axiomatic system. Part of the idea is motivated by trying to develop a theoretical framework that does not preclude the concept of state or property for the system (which is essentially assumed).

The Foulis-Randall theory focuses on test space, i.e., collections $\mathfrak{T}$ of overlapping experimental tests. Letting $X = \bigcup \mathfrak{T}$ be the outcome space of $\mathfrak{T}$, a statistical state on $\mathfrak{T}$ is defined by a function $\omega : X \to [0, 1]$ such that

$$\sum_{x \in T} \omega(x) = 1, \tag{36}$$

for any test $T \in \mathfrak{T}$. A variety of algebraic, analytic, and order-theoretic objects can be attached to any test space $\mathfrak{T}$, each serving as some particular form of logic. If $\mathfrak{T}$ is algebraic, one can construct from the events of $\mathfrak{T}$ a well-behaved ordered partial algebraic structure $\Pi(\mathfrak{T})$ called an orthoalgebra, which have a natural generalization to orthomodular partially ordered sets (of which the projection lattice is an example).

An advantage of this approach is that test spaces are often much easier to analyze and manipulate than their associated logics. They also have the heuristic advantage that the operational definition is readily apparent, since test spaces just correspond to particular experiments. Moreover, if $\mathfrak{T}$ is algebraic, there exists a canonical order-preserving map $L \to \mathcal{L}$ from the logic L of $\mathfrak{T}$ into the property lattice $\mathcal{L}$ associated with any structure $(\mathfrak{T}, \Sigma)$ over $\mathfrak{T}$. In both classical and quantum mechanics, the map is an isomorphism so that L inherits completeness from the lattice while $\mathcal{L}$ derives orthocomplementation and orthomodularity from the ordering relation (implication) of the logic. It is mostly taken for granted that such an isomorphism is the exception rather than the rule. The tendency to identify L with $\mathcal{L}$ is something that has caused a great deal of unnecessary confusion in discussions of quantum foundations, especially in quantum logical affairs.

In 1982, Foulis and Randall collaborated with Piron to develop a comprehensive realist-operationalist framework of quantum theory, establishing the so-called 'Geneva school' approach to quantum physics. The Geneva school argues that the realistic view implicit in classical physics does not necessarily have to be abandoned to accommodate some of the less intuitive concepts of quantum mechanics. Rather, one should instead give up on the presupposition that any set of experimental test possesses some common refinement (in other words, that experiments are always compatible with other experiments). This approach in no way excludes the notion of physical systems existing exterior to an observer, nor does it imply that the properties of such systems depend on what the observer knows about the system. Aside from the previously defined statistical state by Foulis and Randall, the realist-operational version also introduces the notion of a realistic state represented by some type of subset of X called the support, representing all possible outcomes in that state.

Thus, generally speaking, the Foulis-Randall-Piron formalism is primarily concerned with making a sharp distinction between the event calculus or the operational logic and the property lattice which it represents. In Hilbert space, these two mathematical structures are isomorphic, and all of these are isomorphic to the lattice of closed linear subspaces of Hilbert space. However, any axiomatic system that tries to reproduce the projection lattice without assuming Hilbert spaces in advance should carefully distinguish between event and propositional lattices, which usually have different formal structures.

More recent developments of quantum logic in the last few decades take most of the mathematical aspects of this work further, as the various features of abstract test spaces, orthoalgebras and generalized orthomodular structures were studied as a pure theoretical exercise. These modern algebraic studies involve advanced mathematical notions in areas such as category theory, computational semantics, and non-commutative geometries, which are arguably outside the scope of what is needed here.

5 Hidden variables and quantum logic

According to Allen Stairs [25], it is important to recognize the distinction between what is considered the core of the position of quantum logic and the way in which this core is applied or interpreted. The core of quantum logic, first made clear in the work of Demopoulos and Bub [26], says that the physical world has a logical structure that governs the relations of exclusion, inclusion, compatibility and equivalence among possible events or states of affairs that is manifestly different from what one would expect if they satisfied classical logic.

There is sometimes another claim associated with quantum logic, namely that every physical magnitude has a value in every state, which may be referred to as the value-definiteness of properties of the system. This thesis should be taken to be part of the interpretation of the core, which seems more natural in the context of a deterministic hidden variable theory. The difference is that the denial of any hidden variables is fundamental to quantum logic. Formally, one can even use quantum logical methods to reproduce Bell's lemmas [27], a crucial element of the theorem ruling out local hidden variable theories. A related result by Kochen and Specker examines whether a suitable classical phase space with hidden variables can be constructed for quantum systems such that the measurement statistics are recovered, wherein non-contextual hidden variables are ruled out. Before examining the various quantum logical interpretations of quantum theory, a short digression into what quantum logic has to say regarding the issue of hidden variables in quantum mechanics is covered in this section; in particular, the debate between Jauch and Piron on one side and Bohm and Bub in the other is focused on here. For a concise yet somewhat dated background on this topic, one can take a look at Bell's account [28].

In the classic von Neumann proof of the impossibility of a hidden variable model for quantum mechanics, he used an assumption for any real, linear combination of observables, compatible or not. It states the following: for any pair of observables A, B if $C = xA + yB$, with $x, y \in \mathbb{R}$ then

$$v(C) = x(v(A)) + y(v(B)), \tag{37}$$

where $v(C)$ refers to the eigenvalue of observable C. Bell and Mermin remark that this is a "silly" assumption because when A and B do not commute, they are not simultaneously observable and therefore, there is no reason to insist on such a requirement.

In a paper they published in 1963, Joseph Jauch and Constantin Piron [15] claimed to have come to the same conclusion as von Neumann without using the linearity assumption. Their impossibility proof is based on an analysis of the types of experimental questions that can be appropriately asked in the theory. They considered those physical observables with only two alternatives or possibilities, which may be denoted by 0 or 1, and are represented in quantum theory by projection operators. The outcomes of these binary tests, for example that "observable Q has value q," are called propositions of the system. Propositions are compatible or incompatible depending on whether or not the corresponding measurements can be performed simultaneously. Jauch and Piron proved a theorem in this propositional calculus that if a propositional system admits hidden variables then all propositions are deemed compatible. In this way, hidden variables are ruled out for quantum theory since there is

unequivocal evidence that there are pairs of observables such as position and momentum that can not be determined simultaneously.

In a spirited defense of hidden variables, David Bohm and Jeffrey Bub [29] argued that the Jauch-Piron analysis is fundamentally flawed, since it claims that the existence of incompatible propositions is an "empirical fact". Such incompatible propositions could be taken as necessary inferences from experiment only if it could be established that no other propositions besides those of quantum mechanics are valid descriptions of quantum systems. Thus, Jauch and Piron have set out to assume that orthodox quantum theory is correct formally and requires no extensions, which of course, will lead to the conclusion that no hidden variable can be introduced. In fact, their result is somewhat trivial, since they are merely restating well-known results by Bell, Kochen, and Specker in the language of the projection lattice of a Hilbert space. If there are hidden variables do underlie quantum mechanics, Bohm and Bub contend that it is then possible to express experimental questions without using incompatible propositions; rather, simultaneous descriptions of incompatible observables only imply that determining their values involve incompatible processes of measurement.

Jauch and Piron [30] responded to Bohm and Bub's attack by raising the following points:

(a) The validity of quantum mechanics is not assumed in advance. One merely assumes the lattice structure of yes-no experiments that come directly from experimental facts.

(b) No assumptions are made regarding states being linear functionals on the propositions. Indeed, linearity cannot even be expressed here since addition on propositions is undefined.

(c) No coherence is presumed in the lattice, allowing for valid inferences for systems with superselection rules.

They insist that Bohm and Bub have misrepresented their position, one that actually follows in spirit to what von Neumann tried to do but is done by reducing the restrictions imposed by von Neumann to the minimum required for making valid inferences about potential hidden variable models. What they found was that hidden variables in quantum mechanics suggest that every physically realizable state can be represented as a mixture of dispersion-free (that is, zero uncertainty) states. Physically, this means that if one prepares an ensemble under identical relevant conditions, this ensemble in principle could be treated as composed of sub-ensembles which are dispersion-free in all physical quantities. Jauch and Piron claimed that their main result shows that the existence of hidden variables in this sense would entail some properties of the lattice of propositions which are incompatible with known observational facts, specifically those involving Heisenberg-Robertson uncertainty relations.

Bohm and Bub [31] replied to this by pointing out how Jauch and Piron themselves admit that the lattice is an assumption in their analysis. The lattice structure of propositions, as specifically specified by Jauch and Piron, cannot be a unique and inevitable inference from known facts. Bohm and Bub raised the point that axioms of a theory "stand on a different level from the experimental facts underlying the theory." Axioms are always assumptions from which inferences are drawn about what is observed. If the logical inferences agrees with experimental evidence, then the axioms are confirmed. However, confirming the axiomatic structure in this way never implies that no other set of axioms is possible. What happens specifically in Jauch and Piron's case is that they are trying to answer a question framed in a formal structure which excludes hidden variables at the onset. What one needs is to start with an axiomatic framework that allows for the possibility of hidden variables and explore this system to see whether they are ultimately ruled out. There is a suggestion by Bohm and Bub that the model of Jauch and Piron may be modified to allow hidden variables in this manner although the specific details are not mentioned and will not concern us here.

So what is the verdict? It seems to depend on whether or not one accepts that an abstract lattice structure for experimental propositions (tests or outcomes in a more recent contexts) is an appropriate

starting point for developing quantum theory from foundational concepts. If so, then it seems Jauch and Piron have accomplished something significant, albeit it would have more resounding consequences for hidden variables if shown in a similar but more generalized framework like the Foulis-Randall extension of Piron's axioms. If not, then Bohm and Bub's argument wins the day in convincing fashion, with nothing more to say than to reiterate the idea that Jauch and Piron's proof is a misguided exercise in circularity. In some sense, the debate becomes a question of whether the problem of a hidden variable model for quantum mechanics is fundamentally a question about the logical structure of the theory. In fact, a quantum logician would say that quantum mechanics is really a theory about how logic really applies in the world, and it seems to be a very non-classical one.

6 Interpretations of quantum logic

Foundational studies in quantum mechanics have consisted in large part of attempts to avoid certain paradoxes or anomalies in the theory. Bohr and Reichenbach attempted to avoid some of these seeming absurdities by altering quantum theory's logic rather than any of its specific axioms. Some examples of these paradoxes include the two-slit paradox, the quantum tunneling paradox, the orbital electron paradox, the Schrodinger cat paradox, and the EPR-Bohm paradox. The two slit paradox was discussed in the first section and was used for motivating quantum logic. The tunneling or barrier penetration paradox involves a particle escaping a potential well despite the fact that its total energy is less than the potential barrier. The orbital electron paradox is due to Heisenberg and refers to the how the probability to find an electron an arbitrary distance from the nucleus of any atom is always nonzero. The Schrodinger cat is the famous example in Schrodinger's 1935 paper [32], where the issue about how to explain the non-existence of simultaneously dead and alive cats. The EPR-Bohm paradox has to do with the intertwined properties of entangled particles, such that measuring the value of a certain property of one particle determines precisely the value of the same property for the other particle no matter how far apart these particles may be.

The usual solution to these problems is to adopt the Copenhagen interpretation, usually attributed to Heisenberg and Bohr, which would say that the interaction between objects and measuring instruments sets an absolute limit to what can be said about objects independent of the observation. Heisenberg remarked that the concept of the probability function does not allow a description of what happens between observations. Any attempt to find such a description would lead to contradictions. For example, there is no paradox for orbital electrons because the argument for it demands reference to the energy at a particular position but such would require the simultaneous observation of the position and momentum, which never happens. If the result of measuring a quantity cannot be predicted with certainty, no statement about its value is true or even meaningful.

One of the main objections by a quantum logicians to the Copenhagen interpretation is that the theory embodies a certain formal awkwardness—specifically that well-formedness of a logical proposition is not a purely syntactic property. For reasons such as this, Reichenbach proposed to admit sentences about measured quantities as meaningful but to ascribe to them a third truth-value, which he called 'indeterminacy'. Putnam offered a more comprehensive program that outlined some of the most ambitious claims made on behalf of quantum logic: a realist interpretation of empirical quantum logic. More recent interpretations have backed down from a strong realist thinking and have adopted an operational approach, where 'properties' and 'states' of systems are mostly treated in an empirical manner in terms of experimental tests. These various interpretations of quantum logic are explored in this section.

6.1 Reichenbach's three-valued logic

Reichenbach's trivalent quantum logic (in contrast with bivalent Boolean logic) is fundamentally a semantically motivated interpretation of quantum theory that provides a rather simple solution to the problem of meaningless statements. The starting point of Reichenbach's consideration was Heisenberg's principle of indeterminacy, which implies that a physical property has no definite value whenever its complementary variable has been measured. As a logical empiricist, Reichenbach subscribed to the ideal of a scientific language not containing any meaningless statements at all. He proposed to reserve the label 'meaningless' to statements about quantities unmeasurable in every physical situation and call statements which may have been true or false in a different circumstance (for example, for positions when the momentum is not measured) indeterminate.

The truth value 'indeterminate' is considered by harsh critics to be a bastard of sense and nonsense, mostly because they feel it is not sufficiently well-defined [33]. However, this accusation is completely unfounded: a simple example will suffice to show how indeterminate statements occur all time. Borrowing a proposition from arithmetic, let us consider the conditional definition for division of real numbers,

Definition 12. *Given $a, b \in \mathbb{R}$, there is an operation called division such that when $b \neq 0$*

$$\frac{a}{b} = c \Leftrightarrow a = b \cdot c, \tag{38}$$

where $\cdot$ refers to the multiplication of two real numbers and $c \in \mathbb{R}$.

The value of $\frac{a}{b}$ is said to be indeterminate [4] when the initial condition is not satisfied, i.e. $b = 0$. Thus, an uncontroversial way to get indeterminate statements is from conditional definitions, something that is encountered all the time in mathematics. The philosopher Ulrich Blau even claims that the informal logic of our normal language is more adequately reconstructed as a three-valued logic [34].

However, it must be said that three-valued logical operations do not fit properly into truth tables in the same manner that Boolean logic easily would. The adequate way to define them will have to use conditional definitions. Obtaining an 'indeterminate' truth value for a logical operation just means it is undefined for the propositions involved.

In Reichenbach's logic, the set $\mathcal{M}$ of measurement propositions is not closed under most of the logical connectives, that is, not all propositions one can write down using operators $\wedge, \vee, \neg$, for example, are necessarily permissible propositions. This is a direct consequence of the fact that most logical sentences will involve conditional definitions, in particular those statements pertaining to measurement situations with incompatible observables.

Reichenbach's truth functional theory was criticized by many because while von Neumann's lattice exhibit some of the structure inherent to the conceptual framework of quantum theory, namely the orthocomplemented, weakly modular lattice for measurement propositions, this important structure can't be discovered in Reichenbach's quantum logic. However, this is again not the case. When considering the set $\mathcal{M}$ as a lattice, one just requires additional postulates to recover an orthocomplemented, weakly modular structure. Moreover, unlike other axiomatic frameworks, the postulates require no extra-logical notions. (For example, Mackey's formulation requires the concept of probability, which is strictly not a syntactic logical element.) That none is needed here demonstrates the strength of Reichenbach's theory.

Omitting the technical specifics, the axioms needed pertain to contradiction ($A \wedge \neg A$), diametrical negation ($-A \in \mathcal{M}$) (which is different from complete negation $\neg A$ and cyclical negation $\sim A$),

[4] In most contexts, the value of $\frac{a}{b}$ can be safely set to be infinity, but if say $a = 0$, then clearly $\frac{a}{b}$ is indeterminate.

three-valued conjunction $\vee$, weak modularity, and the law of contraposition for diametrical negation and alternative implication $(A \to B) \Leftrightarrow (-B \to -A)$. For more details on how these trivalent logical operations are defined, the interested reader can refer either to page 151 of Reichenbach's textbook [11] or to Kamlah's paper [33]. Once these axioms are introduced, the set of measurement propositions becomes an orthomodular lattice in the natural way—if $A \to B$ is read as the lattice theoretical implication. Reichenbach's logic is an analytical theory, which may be derived from the non-classical semantic conventions for its operations. It differs from Boolean logic only in how it accounts for propositions pertaining to incompatible observables, which are deemed meaningful under certain empirical conditions and undefined under others.

In the usual Hilbert space picture of quantum mechanics, a statement is considered indeterminate whenever it concerns a property for which the state in question is not one of its eigenstates. Although this may seem to render indeterminate statements about observables in a very large class of practical measurement situations, Putnam [36] explains that indeterminate refers only to statements that are neither accepted nor rejected at the present time. He insists that truth value of any logical statement is necessarily an epistemic predicate, which means that it is always relative to evidence at a particular time. Upon verification or falsification, an indeterminate claim goes from a 'state of limbo' into being true or false. This is the case even for incompatible observables because the process of verification will operationally lead to a determination of one of the two non-commuting variables. There is no need to worry about simultaneous measurements of properties like position and momentum because they are not observed empirically. Hence, Putnam insists in the empirical nature of logic in this precise manner.

6.2 Putnam's realism

Kochen and Specker's original proof about contextuality was based on taking at face value the way quantum mechanics appears to represent the functional relationships among magnitudes. Quantum logicians believe that treating propositions associated with the same subspace as equivalent is the most natural way of understanding quantum theory. In discussions on quantum logic in Redhead's book [24], he is explicitly concerned with a variety of realism according to which every magnitude has a value in every state (which Stairs called value-definiteness), essentially the kind of realism espoused by Einstein.

Quite independent of his remarks about Reichenbach's trivalent logic, or any manifestation of quantum logic for that matter, Putnam takes into consideration this 'value-definiteness criterion' of reality and establishes how quantum logic should be properly interpreted realistically. His position is succinctly expressed in the following proportion, what he called the 'heart of quantum logical interpretation [37]':

$$\text{Geometry} : \text{Relativity} = \text{Logic} : \text{Quantum Mechanics}. \tag{39}$$

Putnam justifies this by commenting on how, for instance, relativity can in principle be expressed in pure Euclidean geometric terms. He says,"one can stick to Euclidean geometry provided one is willing to pay a price, the acceptance of causal anomalies—mysterious forces, instantaneous actions at a distance, and so on." In an analogous manner, advocates of quantum logic will assert that classical logic is equally valid for physics provided one is willing to accept certain paradoxes—for example, the Heisenberg cut between the quantum system and the classical measuring apparatus.

A prominent feature of Putnam's realist picture of quantum logic is the belief that the meaning of the logic itself is decided empirically, as opposed to being fully determined by formal relationships between the sets used to represent the propositions. Part of the quantum logical view is that it is a factual, synthetic, empirical matter of deciding which physical situations are describable by the

implication defined on the projection lattice $\mathbb{P}(\mathcal{H})$.

In this version of realism, the overall world-view considers physical processes to involve interactions between particles. Each of these particles have a momentum or position but one must not conclude that each of these particles has a position and momentum simultaneously. This is an instance of the distributive law and is one which fails to hold in quantum logic. A system has many possible state vectors; by right, it has a state vector for each non-degenerate physical observable. However, it is impossible to assign more than one state vector to any system. A physical system consisting of a single particle has a position and it has a momentum. But if the position is known, the momentum can not be determined, and vice-versa, because of the quantum indeterminism that applies to this pair of properties. In this case, only the property whose value is known between any non-commuting pair of observables can be assigned to the system. It must be emphasized that the logic itself does not exactly say how a particular determination of position renders the momentum uncertain.

A quantum logical proposition corresponding to a state vector in Hilbert space is to be understood as representing the logically strongest consistent statement about the system property of interest. To add any other information to such a statement leads either to redundancies or contradictions. For example in classical physics, points of phase space correspond to the logically strongest consistent statements. No extra knowledge about the state is gained by specifying any other kind of information.

Here is a summary of some of the key features of a realist quantum logical view of the world according to Putnam:

(a) Measurement only determines what is already the case; it does not bring to existence the value of the observable measured.

(b) Complementarity is fully retained. The failure of distributivity prohibits the simultaneous existence of certain physical situations but it does not prohibit the determination of objective properties through measurement.

(c) The rules of quantum mechanics can not be supplemented by dynamical laws of the traditional Hamiltonian or Lagrangian kind. The reason is such equations of motion naturally lead to a notion of classical-type phase space for physical states but simultaneous position and momentum values are not allowed by quantum theory.

(d) Probability enters in quantum theory like in classical physics: through consideration of large ensembles of identically prepared systems.

(e) Hilbert spaces in quantum mechanics simply represent various 'logical' spaces; namely, the lattices of experimental propositions relevant to the physical system studied. The lattice structure is a partial Boolean algebra. It is not isomorphic to any Boolean lattice but it can be isomorphic to the lattice of subspaces of a suitable linear space (additional assumptions are necessary to recover the projection lattice $\mathbb{P}(\mathcal{H})$ exactly.)

6.3 Modest quantum logic

Putnam's version of quantum logic calls for a traditional kind of realism, one where Einstein's sufficient condition for reality implies the pre-existence of the value of the physical property being measured. However, Stairs noted that the condition that "all physical observables have classically definite values may impose restrictions on the class of allowable structures in the theory in a way that has nothing to do with realism [25]." It may be possible that if the physical magnitudes obey a non-classical logical structure, they can be made compatible with realism without violating locality, even if value-definiteness of observables is not imposed.

Demopoulos reminds us of an important distinction to be made between correspondence with the truth and coherence among different truths. The minimal condition for truth consists of the correspondence between a proposition and an independent state of affairs. One might hold that a proposition Q is true only if there is an assignment of truth values to all true propositions under which q is assigned the value 'true'. But such a constraint makes the truth of a particular proposition coherent with other propositions. Classically, there will always be a truth functional that assigns all true propositions the value 'true' but what Demopoulos claims is that the logic of quantum theory is telling us that this is an inessential feature of truth [35]. A realist should be committed to correspondence, and not necessarily coherence. It is a mistake to think that coherence is a sufficient condition for truth, although it is also unclear one should deny that coherence with other truths is a necessary condition for truth. To this the quantum logician might respond that even if coherence turns out to be necessary for truth, it must be a quantum logical kind of coherence which one must understand.

A well-known implication of quantum logic is that compound propositions involving the standard logical connectives do not necessarily acquire their truth value from the truth value of their more elementary constituents, i.e. the logic is not truth-functional. This is just another way of saying that the relations of equivalence, inclusion, exclusion, etc. among the possibilities open to quantum systems have a characteristic non-Boolean structure. In this light, Stairs proposed a more modest form of quantum logic where logical relations are understood in terms of truth values rather than say experimental questions or measurement results. He calls it modest quantum logic, contrasting it with Putnam's strong realist version, and its distinguishing feature is that it does not include value-definiteness as a necessary component of realism. Modest quantum logic offers a theoretical framework that provides the same advantages as that of a stochastic hidden variable theory but without introducing the hidden variables themselves [25].

Some key distinctions between Putnam's strong realist and Stairs' modest quantum logic:

(a) **Measurement:** In modest quantum logic, pre-measurement values are indefinite. This gives a special role to measurement. This might sound like the unresolved measurement problem or rather a solution to the problem by introduction of an ad hoc process. But there is a difference between postulating something for which one has no convincing explanation and postulating something without any convincing reason. Measurement and its projection postulate is really the former kind and hence is not as big of a problem as people make it out to be, especially if we believe there is a more fundamental structure to the quantum theoretical formalism.

(b) **Probabilities:** In Putnam's realism, probabilities are epistemic—they result from degrees of belief and correct assignment of probabilities require rational, coherent assignments specified by quantum theory. In Stairs' modest version, probabilities are physical—they represent propensities or dispositions for various outcomes to be realized on an ideal measurement. (Ideal just means that when a certain property is already true for the system then it must yield that outcome exactly, e.g. it is in an eigenstate of some observable.)

(c) **Luder's rule:** In strong quantum logic, Luder's rule represents a probability conditionalization upon acquisition of new information. In modest quantum logic, it is the same sort of conditionalization but any form of knowledge acquisition is denied. What one says is that a certain possibility has been realized and therefore the disposition or propensities get conditioned upon this realization.

(d) **Contextuality:** Both versions of quantum logic avoid contextualism but from slightly different interpretational perspectives, which is not too important to distinguish here.

(e) **Meta-language:** $Q \vee R$ is true does not necessarily imply Q is true or R is true—this is just one of the ways in which the meta-language does not work in the usual classical sense. The

course of actual events is constrained in a non-trivial way by what is possible. Stairs' logic is not the same as Copenhagen interpretation: systems that are in the eigenstate of a magnitude or with indefinite magnitude (which is equivalent to the proposition $Q \vee R$) are considered real, independent of measurement.

6.4 Finkelstein's empirical logic

Starting with the primitive notion of a test or quantum operation, Foulis and Randall formulated the formal scheme of empirical quantum logic, where test spaces are the key mathematical objects. Finkelstein shares much the operationalist mindset in his own formulation of quantum logic [3], but he uses a slightly different but almost synonymous language of effectors (input from the system) and receptors (output from the measuring device)—where tests are considred to be filtration or transmission processes acting on the state of the system. The major difference between Foulis-Randall-Piron and Finkelstein is that while the former attach a realist interpretation to their framework, the latter insists on a purely empiricist reading. In fact, Finkelstein's position appears to be not just instrumentalist but also strongly anti-realist when he said, " quantum theory denies the existence of an absolute reality [38]". Furthermore, the Foulis-Randall-Piron scheme has questions and propositions which strictly speaking have separate but isomorphic algebraic structures. In the case of Finkelstein, he only has the lattice of experimental tests. He makes use of terms like properties and states to make connections with the standard language of quantum theory but points out that these are derived notions from outcomes of certain operations. Since the realist operational scheme has already been covered in some detail in the last section, our discussion here will concentrate on Finkelstein's version of empirical logic.

Finkelstein's operational approach tries to present the principles of classical and quantum logic in a unified framework using the appropriate operational notions of properties and states. His operational theory expresses experimental situations in the language of complex matrix algebra. Properties are determined by performing experiments, which involve filtrations (measurements) or transmissions (reversible dynamics). States are defined in terms of ensembles of identical preparations. The classical states has an absolute sense but a quantum one has states relative only to an operational frame.

In the standard prototype classical experiment of a coin with heads and tails (fig. (3) shows its classical phase space), all matrices for filtrations F and transmissions T are binary. In general, for classical systems that are completely known (at least those that are not mixtures of different classical states) filtrations are matrices with a single 1 in the diagonal and zeroes elsewhere (diagonal projectors), while transmissions are permutation matrices, and states are filtrations with unit trace. All filtrations commute–the classical algebra of classes is commutative. One may call a vector a state vector whenever the projection operator corresponding to the vector is a state in this language. It also possible to define a classical principle in this language: any two state vectors for a classical system are orthogonal or parallel. The states and operations are described in the same matrix algebra but the states generate a distinguished commutative subalgebra as a result of the classical principle.

The main change in quantum theory is the quantum principle: every non-zero vector is a state vector. In the quantum case, the filtrations need not be binary matrices but may have any complex number as long as F is a projection matrix while transmissions refer to any unitary matrix. States are still filtrations with unit trace. However, the filtrations no longer commute and no longer make a natural Boolean algebra. Finkelstein reminds us that state vectors in this formalism represent our actions or the ensemble our actions produce or accept, not a list of properties that specify the system.

In quantum physics, different states need not exclude each other. Instead one introduces a quantum frame, a set of states $\{\Psi_n\}$ that sum to identity. These are actions that are available to an experimenter as long as the states commute. Each frame $\{\Psi_n\}$ has its state variable $S = \sum n\Psi_n$. According

to Finkelstein, S defines a multichannel analysis that always outputs one of the frame states. He is basically just describing what a positive-operator valued measurement (POVM) would be in his empirical scheme. Furthermore, a physical observable V is defined by a spectral resolution $\{P_n\}$ (that is, involving a set of orthogonal projectors) of the identity and the assignment of a value of the observable to each projector in the family. The observable is mathematically expressed by a normal operator $V = \sum v_n P_n$, which is just the spectral decomposition of the hermitian operator associated with V.

Finkelstein's theory is a version of empirical logic because statements about states and properties of the system include only logical propositions one can make from experiments or tests for which there is a well-defined procedure, either a filtration or transmission process in his language. The state vector has no ontic status; in fact, he has a definite sense in saying that the state vector has no physical reality: asking "what is the state vector?" is not an admissible question about properties of a system. However, it is a legitimate question to ask,"is the quantum system in the state ψ?" since the logical framework allows for an answer to such a question. Thus, the state vector must be regarded as a syntactical element employed in describing possible physical properties of a system.

6.5 Dialogic approach

It has been so far that interpreting quantum logic in physical terms involves providing a suitable physical interpretation for the possible logical propositions of the system, whether the primitive notions of the logic take properties or tests as the fundamental concept. In all the quantum logical interpretations discussed in this section, it has been implicitly assumed that the algebraic structure of propositions correspond to a lattice L_q of Hilbert subspaces, where the subspaces can be thought of as analogous to subsets of phase space in the classical Boolean lattice. The similarity between the set-theoretic operations and the logical operations allows us to interpret the lattice of subspaces as a propositional calculus, referred to as the quantum logic. However, propositional logics do not exclusively derive their meaning via set-theoretic ideas. Mittelstaedt [39] provides us with an elegant example of operational semantics that interprets a lattice as a logical calculus that is also suitable for quantum mechanics.

In the lattice theory of partially ordered sets, there are many kinds of lattices that lead to various propositional calculi. A Boolean lattire L_B of propositions corresponds to the calculus of classical logic and an implicative lattice L_i leads to a model for Brouwer's intuitionistic logic, which is primarily characterized by its rejection of the law of excluded middle. (Roughly speaking, the law of excluded middle states that either A or $\neg A$ must true. Brouwer argues that such is not necessarily the case in for statements involving infinite collections.)

The important question is regarding the semantic interpretation of L_q. For the Boolean lattice L_B, it is easy to interpret using a two-valued truth function that assigns 'true' or 'false' to propositions. It has been shown by Gleason, Kamber, Kochen and Specker that neither a two-valued function nor a generalized truth function of any sort does exist on the lattice L_q. In the so-called intuitionistic logic, the lattice L_i can't be interpreted by truth values. However, an implicative lattice can be considered as a logical calculus if one uses the more general, operational method called the dialogic approach. A generalization of the dialogic method can be used to interpret a lattice L_{eq} which is isomorphic to L_q if one adds to its axioms *tertium non datur*, the law of excluded middle. The aforementioned modification of the operational dialogic method has to do with the treatment of incommensurable quantum mechanical propositions, and the modified propositional calculus shall be called 'effective quantum logic'.

In a dialogic approach [40] developed by Giles, Stachow, and Mittelstaedt, one employs game-theoretic notions for two competing parties, which are referred to as the verifier and the falsifier. These

Alice	Bob
1. $b \to (a \to b)$	1. proof of b; why $a \to b$?
2. $a \to b$	2. proof of a; why b?
3. b	

Table 1: *An illustration of the dialogic approach to logic. Propositions are questioned and defended in a conversation between verifier (Alice) and falsifier (Bob). The truth value of any initial claim is reduced to the truth value of an elementary proposition.*

two parties engage in back-and-forth exchange of questioning and proof until the conversation is reduced to determining the truth value of an elementary proposition E, in which case the verifier 'wins' if E is true, while the falsifier 'wins' if E is false. Table (1) illustrates by an example how this interplay works between verifier Alice and falsifier Bob. In this example, Alice starts the dialogue with the proposition $b \to (a \to b)$, which she claims to be true. Bob's move is prove that b is true and challenge Alice why this must imply $a \to b$. Alice must then give a proof that $a \to b$ is true. Bob goes on to show a is true and then ask Alice why this must imply b. Alice finishes off the dialogue by trying to prove b, which if she is successful means she wins the argument. Thus, the dialogic method is characterized by such exchanges and decides the truth value of a proposition depending on whether the verifier (true) or falsifier (false) wins. In this particular example, note that Alice does not in fact have to do the last step; she can simply refer to Bob's proof of b in his step (1) to win this dialogue.

Pretty much the same sort of dialogue can be made for the dialogic proof of quantum mechanical propositions. However, the key difference is that in the defense of a proposition, the verifier is not allowed to cite a previously proven proposition by his opponent whenever the compound statement it belongs to involves an incommensurable propositions. Going back to table (1), if a and b were propositions about position and momentum, respectively, Alice would not be allowed to use Bob's proof of (b) to win the argument.

To incorporate such a rule, one has to define precisely what it means for a, b to be commensurable: two propositions a, b are commensurable if the corresponding observables can be measured in an arbitrary sequence on the system of interest without thereby influencing the result of the measurement. In this case, a, b are always commensurable whenever either $a \leq b$ or $b \leq a$ is valid. It can also be shown that statements involving a and $\neg a$ or $a \to b$ are always commensurable independent of b.

The additional restrictions have the consequence that not all classically dialogically provable implications can still be defended successfully in dialog. Propositions which remain defensible after the inclusion of the commensurability condition will be denoted as quantum-dialogically provable, and any statement which are provable quantum dialogically independent of semantic content.

The calculus of effective quantum logic Q_{eff} can be presented as a system of rules with the aid of which all possible quantum logical propositions can be derived from implications of combinations of a few statements. The calculus Q_{eff} is consistent and complete with respect to the class of quantum dialogically provable implications, so that it is isomorphic to the full quantum logic Q of the lattice L_q of Hilbert subspaces when one incorporates into the logical calculus truth values for the propositions (so far we have only discussed the formal syntax of statements with no reference to truth value assignments) and one adds to the rules the implication

$$1 \leq a \vee \neg a, \tag{40}$$

which is just a statement of *tertium non datur*. It is important to note that the weaker assumption of decidable truth values for every proposition must not be confused with the much stronger postulate of the existence of a two-valued truth function that determines what the truth value is. In fact, it has

already been shown that such two-valued function does not exist for quantum mechanical propositions and this has been the may obstacle to the interpretation of L_q as a logical calculus.

The propositional calculus Q of quantum theory was first expressed in terms of a lattice by Birkhoff and von Neumann, and later developments have attempted to reproduce the lattice structure from an axiomatic system that does not assume Hilbert spaces. However, proponents of the dialog method show us that for the lattice-theoretic characterization of Q it is more appropriate to first reformulate the propositional rules in terms of an effective quantum logic Q_{eff} which has a more convenient but equivalent set of rules. This shows that the lattice of quantum mechanical propositions is really an orthocomplemented quasi-implicative lattice L_{qi}, characterized by a set of axioms that leads to quasi-modularity for the lattice. By postulating the law of excluded middle, one finally arrives at the lattice isomorphic to L_q, which is just the projection lattice $\mathbb{P}(\mathcal{H})$.

7 Evaluating quantum logic

Quantum logical interpretations of quantum mechanics have always been driven by physical situations that do not admit a simple classical explanation. The aim of quantum logic is to establish the correct logical structure of the theory such that the algebraic framework dictates the proper reading of the details of paradoxical experiments, whether it refers to truth values, physical properties (propositions of the system), or measurement outcomes (quantum events).

Reichenbach was mainly concerned with the interpretation of the unobservables of quantum mechanics. Unsatisfied with Bohr's dismissal of propositions about incompatible observables as meaningless, he introduced a third truth value called indeterminate. According to Feyerabend, the trivalent logic effectively involves the classical principle of value-definiteness, that is, a physical property always carries a particular value independent of observation. He criticizes Reichenbach's logic because it suggests that quantum laws do not uniformly apply to observable and unobservable situations. Of course, this is somewhat a weak argument since the theory does explicitly distinguish between commuting and non-commuting observables, so at least there is a physically motivated reason for treating them differently. However, Feyerabend makes a valid point when he says that Reichenbach is forced into his logic mainly because he treats value-definiteness as a necessary feature of quantum theory, when it fact it is only required if one sticks to a classical realist picture of the world. Furthermore, our lessons from Bell, Kochen and Specker already tell us that there are major difficulties when trying to interpret certain quantum situations classically. In trivalent logic every statement expressing an anomaly must have an indeterminate truth-value. In general, every quantum mechanical statement simultaneously involving complementary variables can only possess indeterminate truth-values, which isn't exactly more enticing than Bohr's suggestion that they be considered meaningless.

Reichenbach [11] provided several specific arguments in support of his three-valued logic, which are refuted accordingly:

(a) It should not be the case that statements are meaningful only if observations are made to verify them.

Refutation: While it is true that classical properties can only be assigned to a system when particular conditions are fulfilled, these conditions doesn't necessitate an observer. Classically, it is sufficient to determine a set of constraints that a system adheres to that naturally localizes its 'state' inside some region of classical phase-space, as small as desired.

(b) Reichenbach claims that if it meaningless to ask which slit a photon passes through in a double slit experiment if one doesn't look, then it is equally meaningless to say that it passed through one of the slits.

Refutation: This is obviously false since when we see an interference pattern on the screen, it is still legitimate to say that the photons passed through the slits and not through the walls. What is unclear is how it exactly goes through the slits, precisely whether they go to one or the other slit or if they somehow split and pass through both slits.

(c) Reichenbach says that in his interpretation, all statements are true or false, never indeterminate. **Refutation:** This is a puzzling claim since it is pretty obvious that statements about complementary variables such as position and momentum of a single system seems inevitably indeterminate in his logical scheme. (In fairness to Reichenbach, he might have only meant that statements about incompatible observables will still be meaningful but will neither be true nor false. Nonetheless, it still seems an ad hoc modification that just artificially gives meaning to Bohr's meaningless statements without providing any understanding of how incompatible propositions correspond to reality.)

Reichenbach's version of quantum mechanics is partly based upon the projection postulate, which states a reduction of the wave packet of the Schrodinger equation upon measurement. Such discontinuous, unpredictable transformation of the state vector describes a peculiar sort of interaction which is not governed by all other dynamics processes associated with unitary transformations. However, projection is something that an experimenter chooses to do, and should therefore not be expected to have a counterpart in the dynamical equation of motion of quantum systems.

One may think of measurement without the post-selection, so that there is no projection collapse but this is still not a general measurement description. In fact, the mixing effect of a measurement without post-selection is derivable special case of basic postulates in quantum theory. If one assumes that the state at a time $t_2 > t_1$ is uniquely determined by $t_2 - t_1$, then it is possible to show, with linearity and continuity assumptions that there is an operator H such that for pure states

$$\psi(t_2) = e^{-i(t_2-t_1)H}\psi(t_1) \tag{41}$$

so that the evolution depends only on the time difference. This is a reasonable assumption for an isolated system but not really for an open one. Measurements do not conflict with the Schrodinger equation because they do not satisfy the conditions in which it applies—there is no corresponding Hamiltonian for the transformation it describes.

For Reichenbach's theory, he requires the uncertainty principle to forbid simultaneous measurements of incompatible observables. However, Park, Margenau, and Ballentine have all argued correctly that interpreting uncertainty relations involves measurements of ensembles of identically prepared systems and not the precision of single measurements (which is what Heisenberg seemed to believe in, especially with his electron microscope example).

Another problem with Reichenbach's logic is his unsatisfactory attempt to reconcile three-valued logic with the dichotomy of truth values at the macroscopic level, where he invokes the ignorance interpretation of mixtures to argue that macroscopic states are really in some pure state but we just don't which one among the mixture. Of course this doesn't work because a quantum mixture has no unique pure-state decomposition.

Our rebuke has so far been directed exclusively at Reichenbach's logic but there are also some critical remarks to be said about the other quantum logical interpretations. In the case of Putnam's realism, the problem is how to reconcile the realist view of properties and the quantum logical structure. Michael Dummett [43] argued that for Putnam's realism to work, he requires distributivity to hold. The argument has to do with how the traditional brand of realism (that is, in the sense of Einstein) is often associated with bivalent true values—its corresponding logic contains propositions that are either

true or false. Without Boolean logic, it is unclear how Putnam intends to make his realist version to work, since distributivity is known to fail in quantum theory. Hence, Putnam cannot embrace his brand of realism without embracing classical logic, which makes his endorsement of a realist quantum logic a hopeless cause. It is worth noting that Putnam has long since backed away from this point of view [24]; in particular, he no longer thinks that the principle of value-definiteness, a distinguishing feature of classical realism in attributing physical properties to a system, is necessary for realism.

With regard to Piron's question-proposition system, several deficiencies were pointed out by Hadjisavvas and Thieffine in a couple of papers [44, 45] published in the early 1980s. In particular, they questioned the validity of one of the axioms asserting the existence of any 'product' proposition $a \wedge b$, which they argue is semantically undefinable. They pose a challenge for Piron's axioms by asking how it would explain a series of experiments that involve the product proposition. If their contention holds, it brings into question the relevance of the system as a syntactical scheme.

Foulis and Randall [46] come in defense of Piron's axioms, pointing out that Hadjisavvas and Thieffine are confused about the semantics because they fail to distinguish between properties of a physical system and operationally testable propositions about the system. Each pair of questions $\{x, \neg x\}$ corresponds to a physical operation. The equivalence class of questions $[x]$ corresponds to an assertion about the state of the system that does not necessarily admit to an operational test. The class $[x]$ is regarded as a property of the system which is accessible for some tests x such that securing x or $\neg x$ tells you that the value of the property $[x]$. If one prefers, x corresponds to an operationally testable proposition while $[x]$ is the set of propositions (not all necessarily testable) that give the same value to the property of the system in question. Recall that in standard quantum mechanics, it is presumed, modulo superselection rules, that for every projection operator E, the pair $\{E, 1-E\}$ corresponds to a physical operation or experiment. These sets of experimental tests for propositions of the system that make the connection to projection lattice $\mathbb{P}(\mathcal{H})$.

On a somewhat related note, Bub and Demopoulos [26] argue that Finkelstein's "operational logic" on a 'lattice of experimental tests' is a position excluded by the Kochen-Specker theorem although it is not unclear how this is so. However, note that Piron's axiomatic framework is different from Finkelstein's operational model because the former considers both questions and propositions as necessary elements of the logic, whereas the latter regards tests as entirely sufficient. This is a significant distinction because if Bub and Demopoulos are indeed correct, it shows that not only does one have to distinguish between properties and tests to check the value of those properties, one in fact needs both lattice structures for a complete logico-algebraic description of quantum mechanics. To deny the relevance of one or the other leads to some inconsistency with how certain physical scenarios (especially those involving incompatible observables) have to be interpreted. There are, however, contrary opinions to this matter; in particular, Hughes points out how all the various quantum logical approaches converge to the same formal structures—a partial Boolean algebra that makes the logic of property ascription to quantum systems inevitably nonclassical. He maintains that perhaps this suggests that the issue is not whether any of these interpretations is correct. Rather "it is (a choice) between adopting a deviant logic and eschewing the notion of a property [13]." Maybe it is more appropriate to consider quantum logic as, in Bub's terms, "a non-Boolean possibility structure for quantum events". Of course, as of now, this whole business of how quantum logic works best, is a completely undecided matter.

8 Concluding remarks

In standard quantum mechanics, there are several experimental scenarios that call into question how states or properties of systems should be interpreted in a quantum mechanical context. The big

problem is that classical intuition no longer applies; in fact, to insist on a completely classical logic for experimental propositions leads to anomalies or inconsistencies, especially for situations that involve simultaneous measurements of non-commuting physical magnitudes. Bohr offered a simple solution: since knowing say the position and momentum of a quantum system precisely at the same time is excluded by Heisenberg indeterminism, then one should consider such statements as meaningless. Often called the Copenhagen interpretation, it provides a way of thinking about quantum theory that is good enough for making sense of most practical experiments in quantum mechanics. However, the interpretation leaves much to be desired because of the uncanny success of quantum mechanics as a fundamental theory of nature. How can a theory that works so impressively leave some physical situations unexplained?

Quantum logic is an attempt to bridge the gap in understanding quantum theory, and formally establish the connection between what is observed when measuring quantum systems and the states and properties that can be correctly attributed to such systems. Quantum logic tries to achieve this by looking at the logical and algebraic structure of the theory and examining how the syntax of the logic translates into physically meaningful statements. The algebraic structure has been known for a long time, ever since the work of Birkhoff and von Neumann showed that the quantum propositions form a lattice of closed linear subspaces of Hilbert space. Part of the aims of quantum logical approaches is to show how to reconstruct this lattice from an appropriate set of axioms, without having to postulate that the structure involves Hilbert spaces—giving us the formulations of Mackey, Jauch, Piron, Foulis, Randall, among others.

Probably more important than rediscovering the algebraic structure of quantum theory is the delineation of a self-consistent picture of the quantum world that explains complementarity and entanglement, without invoking 'unphysical' effects such as "spooky actions at a distance" or smudged particles that 'collapse' with measurement. Reichenbach's trivalent logic proposes that the nonclassical lattice structure of quantum mechanical propositions more naturally admits a three-valued truth functional system such that Bohr's meaningless statements (merely) become indeterminate ones. He was obviously motivated more by a dissatisfaction with the Copenhagen position than with reproducing the projection lattice of Birkhoff and von Neumann. Despite the elegant semantics of his logic, there are major difficulties in making it consistent with the classical Boolean logic of commuting observables or effectively macroscopic systems. In particular, it seems that he had no satisfactory way of explaining even the most obvious paradoxes in quantum mechanics.

Putnam's realism embodies the original aim of quantum logic: to reinstate a sort of classical picture of reality by modifying the logical system one uses for evaluating quantum propositions. The main issue is how to reconcile the idea of pre-existing values for properties with quantum mechanical situations that seem to deny this. Proponents of a realist quantum logic have long since abandoned the necessity of definite values for properties in this Einsteinian sense. Such a formal scheme was given by the collaboration among Foulis, Randall and Piron, that drew the attention of quantum logicians to two important lessons: (1) distinguishing between the logical structure L provided by the partially ordered set of subspaces and the lattice structure $\mathcal{L}$ of properties or tests or outcomes and (2) the possibility of considering a logical framework that takes quantum events (i.e., experimental tests and their corresponding outcomes) as the primitive notions as opposed to physical properties of a system. The former is an extremely important distinction, because although in classical and quantum mechanics there is a canonical order-preserving mapping $L \mapsto \mathcal{L}$ which is an isomorphism, in general, their structures are different. Such differences are crucial when one is trying to reconstruct quantum theory from an axiomatic framework of properties and operations. In the latter case, it has opened up a variety of research programs that explore test spaces and outcome spaces for quantum experiments, leading to many important algebraic results of particular value to modern mathematics. The hope is

that the continued study of generalized orthoalgebraic structures will eventually lead to some progress in quantum foundational interpretations as well. The theory of test spaces is important because it drew into attention the obsession of earlier quantum logicians in coming up with an interpretation wherein elements of the logical structure represent physical properties. Perhaps the reason why quantum logic approaches haven't made much headway is because of the insistence on the concept of property attached to a physical system. Maybe the correct way to go is to consider outcomes of experimental tests, which leads to the consideration of quantum events. This is partially embodied in recent programs in quantum foundations such as the convex set framework (where convex set are primitive notions) or modal interpretations of quantum theory (where quantum events are the basic elements).

Despite all the wonderful results since the time of von Neumann, there is one lingering criticism of the entire quantum logical enterprise: it is not clear how much it has actually contributed to our understanding of how quantum theory works at a fundamental level. It seems that looking at the underlying logical structure hasn't really enlightened us on how to properly think about states and properties of system. Despite all the mathematical rigor, even the most fundamental quantum logical results are not really transparent to a non-expert. A large part of it is related to the fact that people take it for granted that classical logic isn't the only logical way to think and reason. Quantum logic does give us an appreciation for how part of the confusion with incompatible observables lies in the fact that our intuition tries to process a nonclassical logical situation with ordinary Boolean logic. A better understanding of the algebraic structure of the theory demands a strong training for non-classical logical thinking, but it doesn't help the cause that it is totally unclear what quantum logic buys you, which is important since the training in logic and advanced mathematics it seems to require is quite a hefty price to pay. Then again, since many foundational issues quantum theory remain unresolved, it is worth pursuing all available avenues of research.

One of the early promises of quantum logic is a consistent realist picture of nature. It now seems that this is no longer available to us, at least not a classical sort of realism where properties always carry values (which is arguably the most desirable sort). It seems to me that anyone who adheres to a quantum logical understanding of quantum mechanics would have to be an empiricist for the most part. In order to justify the assumption of the primacy of a lattice of experimental tests for physical properties of a system, one must argue that the correct starting point for developing a physical theory is to talk only about things that are potentially observable. It seems that to make a stronger realist claim that what one observes exists beforehand, it is necessary to assert that the projection lattice defines some restrictions on valid experimental propositions of a system, basically regarding the associated propositional calculus as tantamount to a natural law. That is a very strong and highly controversial claim albeit one that is still within the realm of possibility. Note, however, that the operational framework that quantum logic provides retains its value for the algebraic analysis of quantum mechanics, even if no suitable interpretation can be attached to the underlying logic. Perhaps quantum logic is just that, a means to understand the algebraic non-Boolean structures of quantum mechanics in a hope that this extra information will lead to a conclusive, cohesive interpretation of quantum theory.

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