Hyperspectral Unmixing from A Convex Analysis and Optimization Perspective

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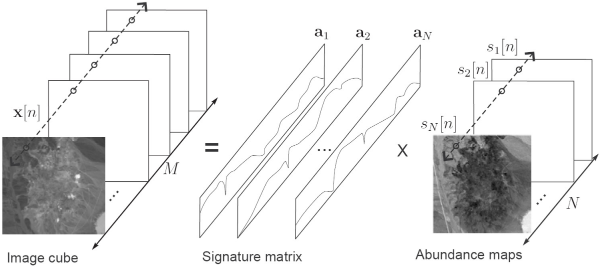
The Theme: Use a convex analysis perspective to view hyperspectral linear unmixing.

- provide formulations & new interpretations for
  - dimension reduction
  - Craig's belief [Craig'94]
  - Winter's belief [Winter'99]
- **Theory:** prove that both Craig's & Winter's beliefs are optimal in the pure-pixel case.
- **Algorithms:** develop convex optimization based approximations for Craig's & Winter's beliefs.

Problem Statement for Hyperspectral Unmixing

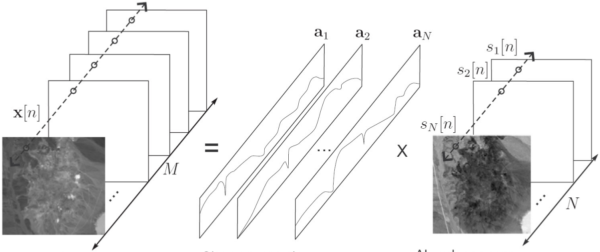


Observed pixel vector: (linear mixing model)

$$\mathbf{x}[n] = \mathbf{A}\mathbf{s}[n] = \sum_{i=1}^{N} s_i[n]\mathbf{a}_i, \qquad n = 1, \ldots, L \tag{1}$$

- $\mathbf{A} = [\ \mathbf{a}_1, \ldots, \mathbf{a}_N\ ] \in \mathbb{R}^{M \times N}$, $\mathbf{a}_i$ is the ith **endmember signature**,
- $\mathbf{s}[n] = [\ s_1[n], \ldots, s_N[n]\ ]^T$ is the **abundance vector** of pixel n,
- M = no. of spectral bands, N = no. of endmember signatures, & L = no. of pixels.

Problem Statement for Hyperspectral Unmixing



Observed pixel vector: (linear mixing model)

$$\mathbf{x}[n] = \mathbf{A}\mathbf{s}[n] = \sum_{i=1}^{N} s_i[n]\mathbf{a}_i, \qquad n = 1, \ldots, L \qquad (2)$$

Some general assumptions:

(A1) (Non-negativity) $s_i[n] \geq 0$ for all i and n.

(A2) (Full-additivity) $\sum_{i=1}^{N} s_i[n] = 1$ for all n.

(A3) $\min\{L, M\} \geq N$ and $\mathbf{a}_1, \ldots, \mathbf{a}_N$ are linearly independent.

Affine Hull

The **affine hull** of $\{\mathbf{a}_1, \ldots, \mathbf{a}_N\} \subset \mathbb{R}^M$ is defined as:

$$\text{aff}\{\mathbf{a}_1, \ldots, \mathbf{a}_N\} = \left\{ \mathbf{x} = \sum_{i=1}^{N} \theta_i \mathbf{a}_i \;\middle|\; \boldsymbol{\theta} \in \mathbb{R}^N, \sum_{i=1}^{N} \theta_i = 1 \right\}.$$

An affine hull can always be represented by

$$\mathcal{A}(\mathbf{C}, \mathbf{d}) \triangleq \left\{ \mathbf{x} = \mathbf{C}\boldsymbol{\alpha} + \mathbf{d} \;\middle|\; \boldsymbol{\alpha} \in \mathbb{R}^P \right\}$$

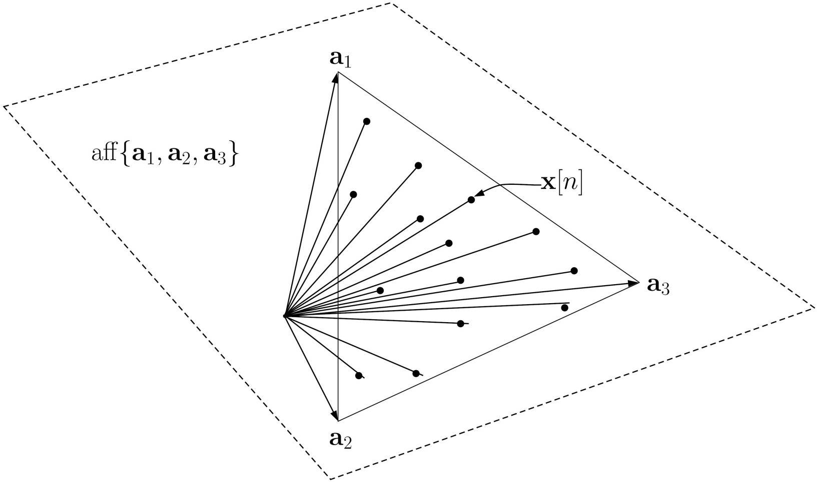
for some $\mathbf{C} \in \mathbb{R}^{N \times P}$, $\mathbf{d} \in \mathbb{R}^N$, & $P \leq N - 1$.

Recall $\mathbf{x}[n] = \sum_{i=1}^{N} s_i[n]\mathbf{a}_i$. Under (A2) and (A3), we have

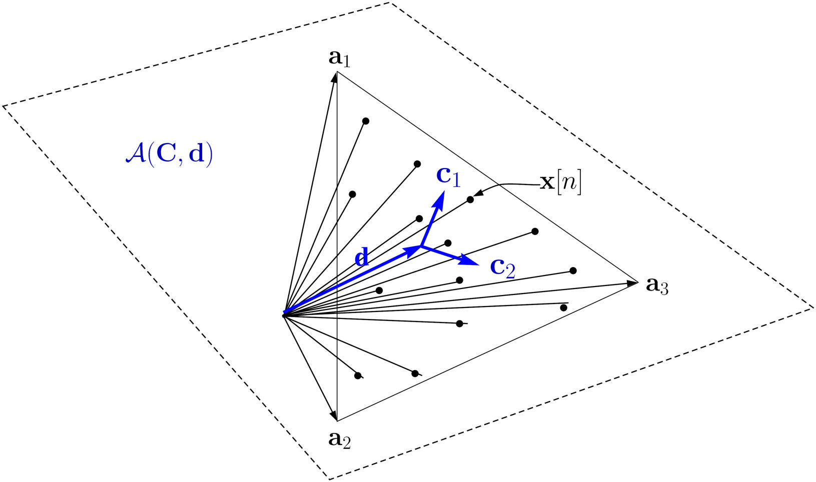
$$\mathbf{x}[n] \in \text{aff}\{\mathbf{a}_1, \ldots, \mathbf{a}_N\} = \mathcal{A}(\mathbf{C}, \mathbf{d}), \qquad \forall n = 1, \ldots, L,$$

with $P = N - 1$.

An Geometry Illustration for $N = 3$



An Geometry Illustration for $N = 3$



How to Get $\mathbf{C}, \mathbf{d}$?

Lemma 1 (Affine set fitting) [Chan'08]

Under (A2) and (A3), we can show that

$$\mathcal{A}(\mathbf{C}, \mathbf{d}) = \text{aff}\{\mathbf{x}[1], \ldots, \mathbf{x}[L]\}.$$

Moreover, $(\mathbf{C}, \mathbf{d})$ can be obtained from $\mathbf{x}[1], \ldots, \mathbf{x}[L]$ by

$$\mathbf{d} = \frac{1}{L}\sum_{n=1}^{L} \mathbf{x}[n], \quad \mathbf{C} = [\boldsymbol{q}_1(\mathbf{U}\mathbf{U}^T), \boldsymbol{q}_2(\mathbf{U}\mathbf{U}^T), \ldots, \boldsymbol{q}_{N-1}(\mathbf{U}\mathbf{U}^T)],$$

where $\mathbf{U} = [\ \mathbf{x}[1] - \mathbf{d}, \ldots, \mathbf{x}[L] - \mathbf{d}\ ] \in \mathbb{R}^{M \times L}$, and $\boldsymbol{q}_i(\mathbf{R})$ denotes the eigenvector associated with the ith principal eigenvalue of $\mathbf{R}$.

- In the presence of noise in the model, Lemma 1 is still optimal in yielding the least squares approximation error in the fitting.

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Relationship to principal component analysis (PCA) [Jolliffe'86]

- The operations of affine set fitting are exactly the same as PCA.
- But affine set fitting has no statistical assumption, it is an outcome of (deterministic) convex geometry.

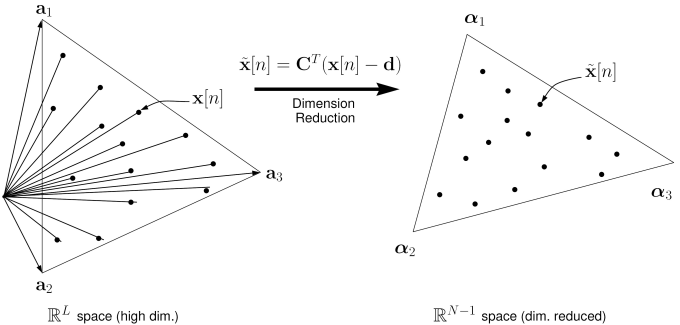
Since $\mathbf{x}[n] \in \mathcal{A}(\mathbf{C}, \mathbf{d})$, its affine representation is

$$\mathbf{x}[n] = \mathbf{C}\tilde{\mathbf{x}}[n] + \mathbf{d} \in \mathbb{R}^M.$$

Then the dimension-reduced pixel $\tilde{\mathbf{x}}[n]$ is given by

$$\tilde{\mathbf{x}}[n] = \mathbf{C}^T(\mathbf{x}[n] - \mathbf{d}) = \sum_{i=1}^{N} s_i[n]\boldsymbol{\alpha}_i \in \mathbb{R}^{N-1},$$

where $\boldsymbol{\alpha}_i = \mathbf{C}^T(\mathbf{a}_i - \mathbf{d})$ is the ith dimension-reduced endmember.



The **convex hull** of $\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_N\} \subset \mathbb{R}^M$ is defined as:

$$\mathrm{conv}\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_N\} = \left\{ \mathbf{x} = \sum_{i=1}^{N} \theta_i \boldsymbol{\alpha}_i \;\middle|\; \boldsymbol{\theta} \succeq \mathbf{0}, \sum_{i=1}^{N} \theta_i = 1 \right\}$$

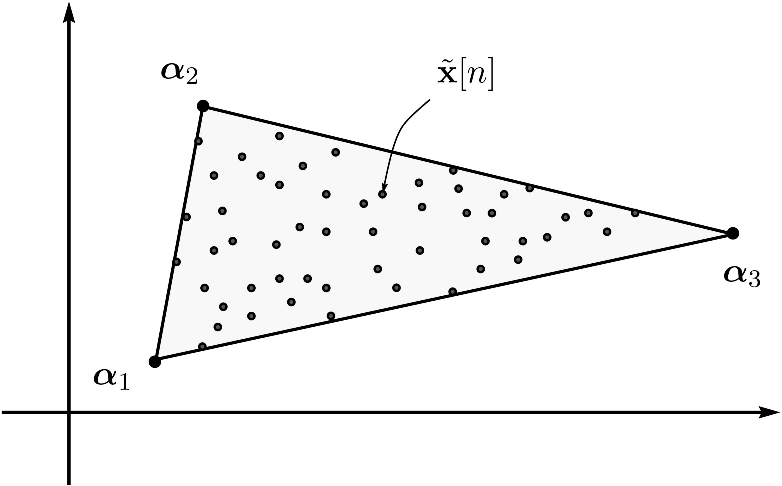
A convex hull $\mathrm{conv}\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_N\} \in \mathbb{R}^M$ is called a **simplex** if $M = N - 1$ & $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_N$ are affinely independent.

Recall $\tilde{\mathbf{x}}[n] = \sum_{i=1}^{N} s_i[n]\boldsymbol{\alpha}_i$, $s_i[n] \geq 0 \forall i, n$, $\sum_{i=1}^{N} s_i[n] = 1$.

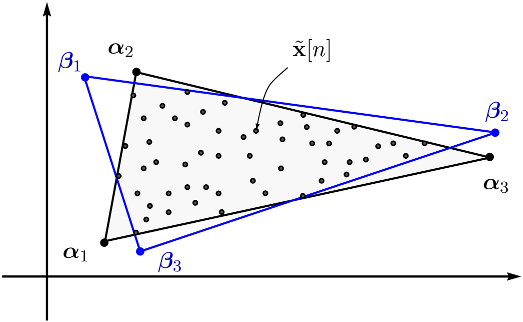
Lemma 2 (Simplex geometry) [Chan'09]

Under (A1), (A2), and (A3), all the $\tilde{\mathbf{x}}[1], \ldots, \tilde{\mathbf{x}}[L]$ are confined by a **simplex** $\mathrm{conv}\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_N\}$:

$$\tilde{\mathbf{x}}[n] \in \mathrm{conv}\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_N\} \subset \mathbb{R}^{N-1}, \ \forall n$$



Question: Could we estimate $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_N$ from $\tilde{\mathbf{x}}[1], \ldots, \tilde{\mathbf{x}}[L]$?



Formulation: Min. Volume Simplex Fitting [Chan'09] [Li-Bioucas'08]

$$\begin{aligned} \min_{\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_N} \quad & V(\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_N) \\ \text{s.t.} \quad & \tilde{\mathbf{x}}[n] \in \text{conv}\{\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_N\}, \ \forall \ n, \end{aligned} \tag{3}$$

where $V(\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_N)$ is the volume of $\text{conv}\{\boldsymbol{\nu}_1,\ldots,\boldsymbol{\nu}_N\}$.

- Inspired by **Craig's belief**: find a minimum-volume simplex enclosing all data points $\tilde{\mathbf{x}}[1],\ldots,\tilde{\mathbf{x}}[L]$. [Craig'94].

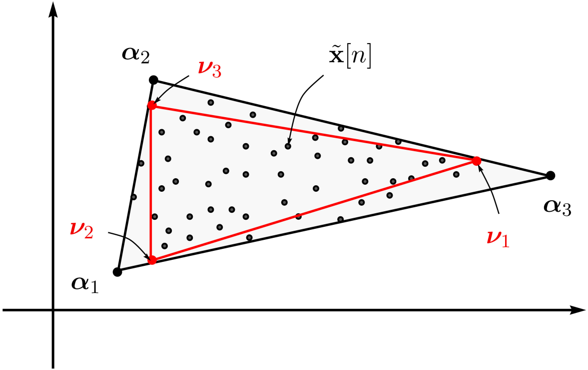
- Craig's belief is sound intuitively. But can we prove some theoretical guarantee of it?
- We prove a sufficient condition for the min. volume simplex problem as follows.

  Pure pixel assumption:

  (A4) For each $i \in \{1, \ldots, N\}$, there exists at least one pixel index ℓ_i such that $\mathbf{x}[\ell_i] = \mathbf{a}_i$.

Theorem 1 (Endmember identifiability of Craig's belief)

Under (A1)-(A4), the globally optimal solution of the min. simplex volume problem is exactly $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_N$, corresponding to the true endmembers $\mathbf{a}_i = \mathbf{C}\boldsymbol{\alpha}_i + \mathbf{d}$.



Formulation: Max. Volume Simplex Fitting

$$\begin{aligned} \max_{\boldsymbol{\nu}_1,\ldots,\boldsymbol{\nu}_N \in \mathbb{R}^{N-1}} \quad & V(\boldsymbol{\nu}_1,\ldots,\boldsymbol{\nu}_N) \\ \text{s.t.} \quad & \boldsymbol{\nu}_i \in \text{conv}\{\tilde{\mathbf{x}}[1],\ldots,\tilde{\mathbf{x}}[L]\},\ \forall\ i, \end{aligned} \tag{4}$$

- Inspired by **Winter's belief**: find a maximum-volume simplex enclosed by $\text{conv}\{\tilde{\mathbf{x}}[1],\ldots,\tilde{\mathbf{x}}[L]\}$ [Winter'99].

Theorem 2 (Endmember identifiability of Winter's belief)

Under (A1)-(A4), the globally optimal solution of max. simplex volume problem is exactly $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_N$, corresponding to the true endmembers $\mathbf{a}_i = \mathbf{C}\boldsymbol{\alpha}_i + \mathbf{d}$.

By Theorem 1 and Theorem 2, we can conclude that

Relation between Craig's and Winter's beliefs

Both the min. & max. simplex volume problems can perfectly identify the endmembers in the pure pixel case.

Solving the Max. Simplex Volume Problem

Formulation: Maximum Volume Simplex Fitting

$$\max_{\substack{\boldsymbol{\nu}_i \in \mathbb{R}^{N-1} \\ \boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_N \in \mathbb{R}^L}} \quad V(\boldsymbol{\nu}_1,\ldots,\boldsymbol{\nu}_N)$$

$$\text{s.t.} \quad \boldsymbol{\nu}_i = \tilde{\mathbf{X}}\boldsymbol{\theta}_i, \quad \boldsymbol{\theta}_i \succeq \mathbf{0}, \quad \mathbf{1}_L^T\boldsymbol{\theta}_i = 1 \ \forall\ i,$$

where $\tilde{\mathbf{X}} = [\ \tilde{\mathbf{x}}[1],\ldots,\tilde{\mathbf{x}}[L]\ ] \in \mathbb{R}^{(N-1)\times L}$.

- The maximum simplex volume problem is a nonconvex optimization problem: The constraints are convex, but the objective

$$V(\boldsymbol{\nu}_1,\ldots,\boldsymbol{\nu}_N) = \left| \det\left(\begin{bmatrix} \boldsymbol{\nu}_1 & \cdots & \boldsymbol{\nu}_N \\ 1 & \cdots & 1 \end{bmatrix} \right) \right| / (N-1)!$$

  is nonconcave.

- Maximizing $V(\boldsymbol{\nu}_1,\ldots,\boldsymbol{\nu}_N)$ w.r.t. each $\boldsymbol{\nu}_i$ is however easy, with convex optimization.

Solving the Max. Simplex Volume Problem

Formulation: Maximum Volume Simplex Fitting

$$\max_{\substack{\boldsymbol{\nu}_i \in \mathbb{R}^{N-1} \\ \boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_N \in \mathbb{R}^L}} \quad V(\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_N)$$

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where $\tilde{\mathbf{X}} = [\ \tilde{\mathbf{x}}[1], \ldots, \tilde{\mathbf{x}}[L]\ ] \in \mathbb{R}^{(N-1)\times L}$.

- By cofactor expansion,

$$V(\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_N) \propto \left| \mathbf{b}_j^T \boldsymbol{\nu}_j + (-1)^{N+j} \det(\boldsymbol{\mathcal{V}}_{Nj}) \right|,$$

  where $\mathbf{b}_j$ & $\boldsymbol{\mathcal{V}}_{ij}$ are variables dependent on $\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_{j-1}, \boldsymbol{\nu}_{j+1}, \ldots, \boldsymbol{\nu}_N$.

- $V(\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_N)$ is absolute affine w.r.t. each $\boldsymbol{\nu}_j$.

- Maximization w.r.t. $\boldsymbol{\nu}_j$ can be globally optimally solved by two linear programs (LPs).

Formulation: Maximum Volume Simplex Fitting

$$\max_{\substack{\boldsymbol{\nu}_i \in \mathbb{R}^{N-1} \\ \boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_N \in \mathbb{R}^L}} \quad V(\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_N)$$

$$\text{s.t.} \quad \boldsymbol{\nu}_i = \tilde{\mathbf{X}}\boldsymbol{\theta}_i, \quad \boldsymbol{\theta}_i \succeq \mathbf{0}, \quad \mathbf{1}_L^T \boldsymbol{\theta}_i = 1 \ \forall \ i,$$

where $\tilde{\mathbf{X}} = [\ \tilde{\mathbf{x}}[1], \ldots, \tilde{\mathbf{x}}[L]\ ] \in \mathbb{R}^{(N-1)\times L}$.

Alternating Method

Repeat

solve the jth partial maximization problem

$$(\hat{\boldsymbol{\nu}}_j, \hat{\boldsymbol{\theta}}_j) := \arg \max_{\boldsymbol{\nu}_j, \boldsymbol{\theta}_j} \quad V(\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_N)$$

$$\text{s.t.} \quad \boldsymbol{\nu}_j = \tilde{\mathbf{X}}\boldsymbol{\theta}_j, \quad \boldsymbol{\theta}_j \succeq \mathbf{0}, \quad \mathbf{1}_L^T \boldsymbol{\theta}_j = 1$$

by two LPs

update $j := (j \text{ modulo } N) + 1$.

Until some stopping rule is satisfied.

Solving the Min. Simplex Volume Problem

Formulation: Minimum Volume Simplex Fitting

$$\begin{aligned} \min_{\substack{\mathbf{B},\ \boldsymbol{\beta}_N, \\ \mathbf{s}'[1],\ldots,\mathbf{s}'[L]}} \quad & |\det(\mathbf{B})| \\ \text{s.t.} \quad & \mathbf{s}'[n] \succeq \mathbf{0},\ \mathbf{1}_{N-1}^T \mathbf{s}'[n] \leq 1, \\ & \tilde{\mathbf{x}}[n] = \boldsymbol{\beta}_N + \mathbf{B}\mathbf{s}'[n],\ \forall\ n = 1, \ldots, L. \end{aligned}$$

Let $\mathbf{H} = \mathbf{B}^{-1} \in \mathbb{R}^{(N-1)\times(N-1)}$ and $\mathbf{g} = \mathbf{B}^{-1}\boldsymbol{\beta}_N \in \mathbb{R}^{N-1}$.
Then, $\mathbf{s}'[n] = \mathbf{B}^{-1}(\tilde{\mathbf{x}}[n] - \boldsymbol{\beta}_N) = \mathbf{H}\tilde{\mathbf{x}}[n] - \mathbf{g}$.

Then the problem can be transformed as [Li-Bioucas'08], [Chan'09]

$$\begin{aligned} \max_{\mathbf{H},\ \mathbf{g}} \quad & |\det(\mathbf{H})| \\ \text{s.t.} \quad & \mathbf{H}\tilde{\mathbf{x}}[n] - \mathbf{g} \succeq \mathbf{0}, \\ & \mathbf{1}_{N-1}^T(\mathbf{H}\tilde{\mathbf{x}}[n] - \mathbf{g}) \leq 1,\ \forall\ n = 1, \cdots, L. \end{aligned} \tag{5}$$

We can use alternating linear programming again!

Computer Simulations

- 100 Monte Carlo runs were performed.
- $\mathbf{x}[n]$: 1000 synthetic pixels $(L = 1000)$.
- $\mathbf{a}_1, \ldots, \mathbf{a}_N$: selected from USGS library $(M = 417)$ [Clark'93].
- $\mathbf{s}[n]$: Dirichlet distribution [Nascimento'05].
- **Performence index:** Root-mean-square spectral angle (error performance measure) is defined as

$$\phi_{en} = \min_{\boldsymbol{\pi} \in \Pi_N} \sqrt{\frac{1}{N} \sum_{i=1}^{N} \left[\arccos\left(\frac{\mathbf{a}_i^T \hat{\mathbf{a}}_{\pi_i}}{\|\mathbf{a}_i\| \|\hat{\mathbf{a}}_{\pi_i}\|} \right) \right]^2}$$

$$\phi_{ab} = \min_{\boldsymbol{\pi} \in \Pi_N} \sqrt{\frac{1}{N} \sum_{i=1}^{N} \left[\arccos\left(\frac{\boldsymbol{s}_i^T \hat{\boldsymbol{s}}_{\pi_i}}{\|\boldsymbol{s}_i\| \|\hat{\boldsymbol{s}}_{\pi_i}\|} \right) \right]^2}$$

where Π_N is the set of all the permutations of $\{1, 2, ..., N\}$.†

† $\boldsymbol{s}_i = [\ s_i[1], \ldots, s_i[L]\ ]^T$ denotes the ith abundance map, and $\hat{\mathbf{a}}_i$ and $\hat{\boldsymbol{s}}_i$ denote the estimated $\mathbf{a}_i$ and $\boldsymbol{s}_i$, respectively.

Simulations for Data with Various Purity Levels

- Six endmembers ($N = 6$) from USGS library were selected.
- We generated seven data sets with different purity levels $\rho = 0.7, 0.75, \ldots, 1$ for performance evaluation.

Purity level

A data set with *purity level* ρ denotes a set of L observed pixels with all the purities $\rho_1, \ldots, \rho_L$ in the range $[\rho - 0.1,\ \rho]$, where

$$\frac{1}{\sqrt{N}} \leq \rho_n = \|\mathbf{s}[n]\| \leq 1$$

is a purity measure for an observed pixel $\mathbf{x}[n] (= \sum_{i=1}^{N} s_i[n]\mathbf{a}_i)$. The closer to unity the value of ρ_n, the more a single endmember $\mathbf{a}_i$ dominates in $\mathbf{x}[n]$.

$\Longrightarrow$ The generated data for $\rho = 1$ includes some highly pure pixels.

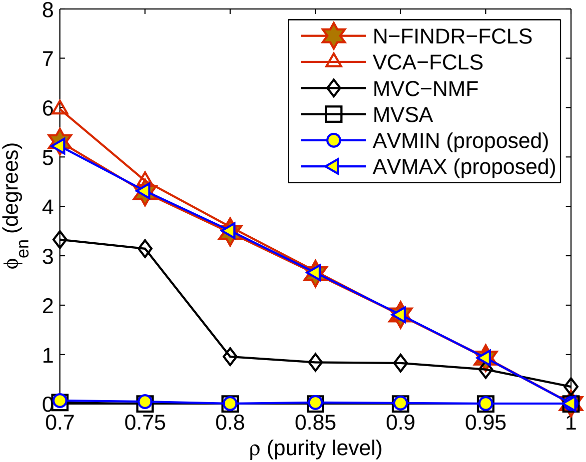

Figure: Simulation results of the endmember estimates obtained by the various algorithms under test for different purity levels (ϕ_{en}).

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[0]VCA: Vertex component analysis [Nascimento'05]

MVC-NMF: Minimum volume constrained nonnegative matrix factorization [Miao'07]

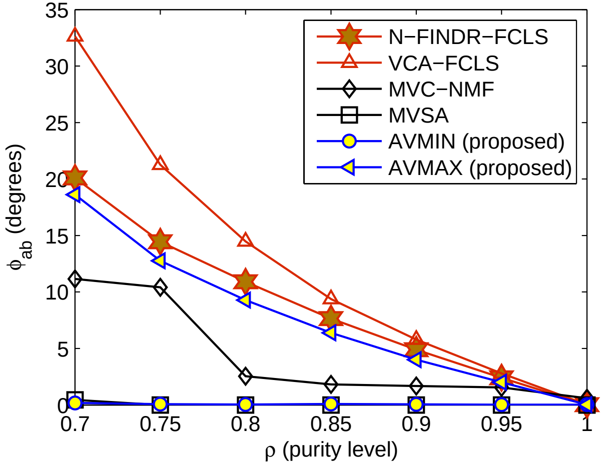

Figure: Simulation results of the abundance estimates obtained by the various algorithms under test for different purity levels (ϕ_{ab}).

[0] MVSA: Minimum volume simplex analysis [Li-Bioucas'08]

Conclusions

- We have provided a convex analysis and optimization perspective to hyperspectral unmixing, from dimension reduction, criteria, to algorithms.
- Open questions arising:
  - theoretical endmember identifiability conditions without pure pixels (positive by simulations, but a tricky analysis problem...)
  - other possible formulations (using determinant as the objective is not the only way out!)

References

Chan'08 T.-H. Chan, W.-K. Ma, C.-Y. Chi, and Y. Wang, "A convex analysis framework for blind separation of non-negative sources," *IEEE Trans. Signal Processing*, vol. 56, no. 10, pp. 5120-5134, Oct. 2008.

Chan'09 T.-H. Chan, C.-Y. Chi, Y.-M. Huang and W.-K. Ma, "A convex analysis based minimum-volume enclosing simplex algorithm for hyperspectral unmixing," in *International Conference on Acoustics, Speech and Signal Processing*, Taipei, Taiwan, April 19-24, 2009, pp. 1089-1092.

Clark'93 R. N. Clark, G. A. Swayze, A. Gallagher, T. V. King, and W. M. Calvin, "The U.S. geological survey digital spectral library: version 1: 0.2 to 3.0 μm," in *U.S. Geol. Surv., Denver, CO.*, 1993, pp. 93-592.

Craig'94 M. D. Craig, "Minimum-volume transforms for remotely sensed data," *IEEE Trans. Geosci. Remote Sens.*, vol. 32, no. 3, pp. 542-552, May 1994.

Jolliffe'86 I. T. Jolliffe, *Principal Component Analysis.* New York: Springer- Verlag, 1986.

Miao'07 L. Miao and H. Qi, "Endmember extraction from highly mixed data using minimum volume constrained nonnegative matrix factorization," *IEEE Trans. Geosci. Remote Sens.*, vol. 45, no. 3, pp. 765–777, Mar. 2007.

Li-Bioucas'08 J. Li and J. Bioucas-Dias, "Minimum volume simplex analysis: A fast algorithm to unmix hyperspectral data," in *Proc. IEEE International Geoscience and Remote Sensing Symposium*, vol. 4, Boston, MA, Aug. 8-12, 2008, pp. 2369-2371.

Nascimento'05 J. M. P. Nascimento and J. M. B. Dias, "Vertex component analysis: A fast algorithm to unmix hyperspectral data," *IEEE Trans. Geosci. Remote Sens.*, vol. 43, no. 4, pp. 898-910, Apr. 2005.

Winter'99 M. E. Winter, "N-findr: An algorithm for fast autonomous spectral end-member determination in hyperspectral data," in *Proc. SPIE Conf. Imaging Spectrometry*, Pasadena, CA, Oct. 1999, pp. 266–275.

Strang'06 G. Strang, *Linear Algebra and Its Application*, 4th ed. CA: Thomson, 2006.

Thank You for Your Attention!