A SHORT PROOF OF THE ERROR BOUND FOR THE TRAPEZOIDAL RULE

YC

The approximation formula for the integral

$$\int_a^b f(t)dt \approx \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{n-1}) + f(x_n)\right).$$

We want to prove the error bound

$$|Error| \leq \frac{K(b-a)^3}{12n^2}$$

provided $|f''(x)| \leq K$. (i.e. f'' has an upper bound K.)

Proof:
First, we divide the interval $[a, b]$ by n-equal subintervals: $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ with $x_i - x_{i-1} = \frac{b-a}{n}$ for any $i = 1, 2, ...n$. Since the error $|Error|$ is the difference of the exact value and the approximation value, we have

$$\begin{aligned}|Error| &= |\sum_{i=1}^{n}\left(\frac{b-a}{n}\right)\left(\frac{f(x_{i-1}) + f(x_i)}{2}\right) - \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} f(t)dt| \\ &= |\sum_{i=1}^{n}\left(\frac{b-a}{n}\left(\frac{f(x_{i-1}) + f(x_i)}{2}\right) - \int_{x_{i-1}}^{x_i} f(t)dt\right)|.\end{aligned}$$

For any i, $0 \leq i \leq n$, we define

$$L_i = \frac{b-a}{2n}\left(\frac{f(x_{i-1}) + f(x_i)}{2}\right) - \int_{x_{i-1}}^{x_i} f(t)dt,$$

and we observe that if the midpoint $c_i = \frac{x_{i-1}+x_i}{2}$,then we have

$$x_i - c_i = c_i - x_{i-1} = \frac{b-a}{n}. \tag{1}$$

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We also have the observation by the integration by parts:

$$\begin{aligned}\int_{x_{i-1}}^{x_i}(t-c_i)f'(t)dt &= \int_{x_{i-1}}^{x_i}(t-c_i)df(t)\\ &= (x_i-c_i)f(x_i)-(x_{i-1}-c_i)f(x_{i-1})-\int_{x_{i-1}}^{x_i}f(t)dt\\ &\overset{by(1)}{=} \frac{b-a}{2n}\left(f(x_{i-1}+f(x_i)\right)-\int_{x_{i-1}}^{x_i}f(t)dt = L_i.\end{aligned}$$

Therefore,

$$L_i = \int_{x_{i-1}}^{x_i}(t-c_i)f'(t)dt. \tag{2}$$

We can use the integration by parts again and the Fundamental Theorem of Calculus,

$$\begin{aligned}L_i &= \int_{x_{i-1}}^{x_i}f'(t)d\frac{(t-c_i)^2}{2}\\ &= \frac{(x_i-c_i)^2}{2}f'(x_i)-\frac{(x_{i-1}-c_i)^2}{2}f'(x_{i-1})-\frac{1}{2}\int_{x_{i-1}}^{x_i}(t-c_i)^2f''(t)dt\\ &\overset{by(1)}{=} \frac{1}{2}\left(\frac{b-a}{2n}\right)^2\left(f'(x_i)-f'(x_{i-1})\right)-\frac{1}{2}\int_{x_{i-1}}^{x_i}(t-c_i)^2f''(t)dt\\ &\overset{Fund.Thm.}{=} \frac{1}{2}\int_{x_{i-1}}^{x_i}f''(t)dt-\frac{1}{2}\int_{x_{i-1}}^{x_i}(t-c_i)^2f''(t)dt\\ &= \frac{1}{2}\int_{x_{i-1}}^{x_i}\left((\frac{b-a}{2n})^2-(t-c_i)^2\right)f''(t)dt.\end{aligned}$$

Thus, if $|f''(t)| \leq K$ on the interval $[a, b]$ we have

$$\begin{aligned}
|Error| &\leq \sum_{i=1}^{n} |Li| \\
&\leq \frac{1}{2} \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} | \left((\frac{b-a}{2n})^2 - (t-c_i)^2 \right) ||f''(t)| dt \\
&\leq \frac{K}{2} \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} (\frac{b-a}{2n})^2 - (t-c_i)^2 dt \\
&= \frac{K}{2} \left((\frac{b-2}{2n})^2 (b-a) - \frac{2n}{3} (\frac{b-a}{2n})^3 \right) \\
&= \frac{K(b-a)^3}{12n^2},
\end{aligned}$$

and we have completed our proof. ■

More questions:

Instead of the assumption for the second derivative $|f''(x)| \leq K$, if we only know the upper bound of the first derivative $|f'(x)|$, we can still have the estimate of the upper bound of the error. However, the accuracy is related to $(\frac{1}{n})$ which is worse than the previous, $(\frac{1}{n^2})$. In fact, suppose $|f'(x)| \leq M$ for all $x \in [a, b]$, by (2),

$$\begin{aligned}
|L_i| &\leq M \int_{x_{i-1}}^{x_i} |t - c_i| dt \\
&= M \left(\int_{x_{i-1}}^{c_i} -t + c_i dt + \int_{c_i}^{x_i} t - c_i dt \right) \qquad = \frac{M}{4} (\frac{b-a}{n})^2.
\end{aligned}$$

This implies

$$|Error| \leq \sum_{i=1}^{n} |L_i| \leq \frac{M(b-a)^2}{4n}.$$